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Track Prospective Mathematics Teachers' Understanding of Special Quadrilaterals: An Exploration of Level 3 of Geometric Thinking

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Abstract: Acknowledging the value of understanding defining and classifying special quadrilaterals for prospective mathematics teachers (PMTs), the present study attempts to track this understanding so that the progress of thinking from Van Hiele's level 2 to level 3 could be theorised. Thus, a bounded case study sample of PMTs, who had graduated and joined the mathematics teacher preparation diploma at the Faculty of Education, Tanta University in Egypt, were selected and requested to (a) define trapezoid, parallelogram, rhombus, rectangle, and square, and (b) represent the relationship among these quadrilaterals. The data were collected and analysed in two cycles. During the first cycle, participants' responses were scrutinised upon Prototype 1; it was developed based on the literature review to describe levels of special quadrilaterals understanding as faulty, partitional-uneconomical, partitional-economical, hierarchical-uneconomical, and hierarchical economical. Similarly, the researchers replicated the same analytical process in the second cycle in order to validate the levels suggested in Prototype 1. Also, some clinical interviews were conducted to confirm the participants' representations of relationships among the defined quadrilaterals. The results enabled advancing the hypothetical Prototype 1 to Prototype 2. Prototype 2 reconceptualised the levels of understanding into faulty, slightly economical, fairly economical, and economical, wherein each level was determined based on (a) the economics of the concept definition and (b) the awareness of relationships among other related definitions to the concept defined (recognising subsets and supersets). These results are prospective for further investigations to sufficiently unpack all sub-levels of geometric thinking embedded in Van Hiele's fixed levels. It also provides basics on proper pedagogical approaches and corresponding interventions to train PMTs effectively teach geometric thinking.

Keywords: Definitions Understanding, Egyptian Education, Geometry Education, Quadrilaterals, Van Hiele Theory

Introduction

On the importance of definitions, De Villiers (1998, p. 249) declared that "(defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalising, specialising, proving, etc.". Definitions allocate properties to mathematical objects, thus, providing a precise meaning for concepts and helping classify them (Alcock & Simpson, 2017; Fujita & Jones, 2007). Furthermore, processes of teaching and learning definitions represent a significant research theme, as described by Sinclair et al. (2016), wherein learners' understanding of defining and classifying shapes determines a meaningful research question to answer. Particularly for mathematics teachers, those who rely on their knowledge of definitions while selecting the pedagogical approaches to teaching these definitions to students (Avcu, 2022; Susanto, 2020). Notably, the importance of emphasising teachers' knowledge stems from what is demonstrated in most literature regarding the strong association between this knowledge, teachers' instructional practices, and students' learning outcomes (Kelcey et al., 2019).

More specifically, the topic of quadrilaterals considers a substantial area of research on reasoning skills that could be developed through recognising geometric figures and exploring and linking their characteristics to form deductive inferences (Okazaki & Fujita, 2007; Van de Walle et al., 2012, as cited in Erdogan & Dur, 2014). Yet, it stays among geometry topics wherein learners experience various difficulties, including defining quadrilaterals and determining relationships among them (Fujita, 2012; Özerem, 2012; Şahin & Başgül, 2020). These difficulties continue even in today's mathematics classrooms because it has multiple origins that, side by side, need to be improved to guarantee the quality of the intended outcomes. They are classified under three categories (a) epistemological (related to students' understanding), (b) ontogenic (students' readiness and maturity), and (c) didactic obstacles (teachers' practices) that are underlined in this study (Brousseau, 2006, as cited in Lutfi et al., 2021).

Defining quadrilaterals requires perceiving necessary and sufficient conditions, which cannot be accomplished before the 2nd level of geometric thinking, according to Van Hiele's theory (Duatepe-Paksu et al., 2012). This could be interpretred within the scope of geometry standards by the National Council of Teachers of Mathematics (NCTM), wherein students should be able to "analyse characteristics and properties of two- and three- dimensional geometric shapes and develop mathematical arguments about geometric relationships" (NCTM, 2000, p. 41). Seriously, if this is expected from pre-k-12 school students, what about prospective teachers who should sustain students in attaining these expectations? Do they possess such knowledge? Do current educational courses help them develop adequate knowledge of basic geometric figures?

That is highlighted in Şahin and Başgül's (2020) study in which, besides the limited representation of concepts in textbooks and the complicated structure of quadrilaterals, the inadequate teachers' knowledge regarding concepts and their focus on the prototypical models remains one reason why students maintain several misconceptions (e.g., failure to expressing the notion of parallelism in cases of rectangles and squares, portray a rectangle as a combination of two squares [Ozkan & Bal, 2017]). As concluded, previous research revealed that most prospective teachers stay unable to (1) precisely define, classify, and determine the minimal characteristics of quadrilaterals; (2) recognise relationships among quadrilaterals; also (3) their level of geometric thinking related to this topic is low (Çontay & Paksu, 2012; Duatepe-Paksu et al., 2012; Erdogan & Dur, 2014; Fujita & Jones, 2007; Fujita, 2012; Kuzniak & Rauscher, 2007; Miller, 2018; Pagiling & Nur'aini, 2022; Pickreign, 2007). For example, in Fujita and Jones's (2007) study regarding how trainee elementary school teachers (aged 18) define special parallelograms and how they perceive relationships among them, it was found that merely 12%, 58.9%, 21.5%, and 38% of participants were able to accurately define trapezoids, parallelograms, rectangles, and squares, respectively. Furthermore, in 2012, Fujita reported that trainee teachers and lower secondary school students are likely to identify quadrilaterals by prototypical examples, which causes difficulties in comprehending the inclusion relationships.

A similar investigation was conducted by Erdogan and Dur (2014), wherein the results revealed that the participants (pre-service primary mathematics teachers) were affected by the prototypical images of quadrilaterals learned during their schooling years and, consequently, they were neither able to use the hierarchical definitions of quadrilaterals nor recognize relationships among them. In detail, although some teachers correctly defined trapezoids, parallelograms, rectangles, rhombuses, and squares, only 40% of those teachers were able to demonstrate hierarchical relationships. On the contrary, the results of Ndlovu (2014) were much better, in which the pre-service teachers expressed a higher Van Hiele level of comprehending geometrical definitions than class inclusion.

Also, Ulger and Broutin (2017) examined third-grade pre-service secondary school mathematics teachers' understanding of quadrilaterals in terms of personal and formal figural concepts. The results indicated that the participants based their definitions on prototype figures and partially expressed the internal relationships between quadrilaterals through these definitions; that is, they had difficulty when understanding the inclusion relations of quadrilaterals. Similar to that were Bharaj and Francis's (2020) investigation; it attempted to determine prospective elementary mathematics teachers' (aged 17-24) conceptions of the characteristics of the quadrilaterals. Accordingly, their findings indicated that the participants seemed to have specific prototypical images of quadrilaterals (e.g., the rectangle defines a figure with two long and two short parallel sides and four 90° angles). Such findings were interpreted by Kabaca (2017), wherein he asserted that the individual's first visual definition of a geometric figure (i.e., personal concept image) considers a principal obstacle to perceiving relationships among quadrilaterals.

Recently, Avcu (2022) explained pre-service middle school mathematics teachers' concept definitions of special quadrilaterals considering the linguistic structure of the Turkish names given to these quadrilaterals, wherein the syntactic, semantic, and lexical structures of names given to quadrilaterals were regarded as possible reasons affecting the defining process. This matches Susanto's (2020) argument on the effect of the language on the definition understanding; as stated, "The terminology in the Indonesian language for rectangle (literally "long square") has a strong effect on teachers' concept image" (p. 3); accordingly, it is difficult for learners to pretend that the squares are rectangles. Again, in the Turkish context, Özdemir and Çekirdekci (2022) conveyed teacher candidates' lack of recognising quadrilaterals by the relationships among them, which resulted from strengthening the edge properties while ignoring the diagonal properties of quadrilaterals; besides, they argued that, generally, contents of undergraduate geometry are not sufficient.

These studies expose that the research on prospective teachers' understanding of quadrilaterals is yet debatable and requires further investigation; hence, the current study endeavours to contribute to this arena. Briefly stated, rather than presenting characteristics of the lack of understanding of quadrilaterals and their reasons, like what is already detailed in the literature, this study embraces

another perspective of exploring the progression of this understanding. Therefore, on one side, the previously defined deficiencies could be assigned to suitable levels of understanding, and on the other side, future activities could be proposed to overcome them.

Research Problem

Although previous research clarified what difficulties prospective teachers encounter when defining quadrilaterals and some possible causes for these difficulties (e.g., personal concept image, lexical ambiguity, strengthening the edge properties), a few of them have regarded levelling teachers' understanding of definitions depending upon their accuracy. Besides, the connection between teachers' definitions and their awareness of relationships among special quadrilaterals is not well expressed throughout the previous studies. Some of these gaps are handled in the current investigation; it aims at uncovering Van Hiele's level 3 of geometric thinking within the context of quadrilaterals. In other words, the progress of thinking from Van Hiele's level 2 to level 3 will be described based on teachers' definitions of special quadrilaterals and their representation of the relationships among them.

According to Van Hiele's model, the most notable theoretical framework for learning geometric figures and their properties (Wang & Kinzel, 2014), learners at level 1 (Visualization/ Recognition) stay influenced by prototype images or visual properties acquired in their early learning (e.g., parallelograms should be inclined, rectangles have two long and two short sides). Then, at level 2 (Descriptive/ Analytical), they are expected to recognise such figures by their properties. At level 3 (Informal deduction/ Rational), which constitutes the beginning of deductive reasoning and definition formation, learners could define figures hierarchically since they could identify interrelationships among them. As reported by Fujita and Jones (2007, p. 5), "learners at van Hiele level 3 are, for example, expected to be able to deduce that a rectangle is a special type of parallelogram by considering definitions and properties of these quadrilaterals". Later at level 4 (Formal deduction), they can construct mathematical proofs employing propositions, axioms, and theorems. Finally, at level 5 (Rigor/ Mathematical), learners could transfer their understanding to other non-Euclidean geometries (Çontay & Paksu, 2012; De Villiers, 1994; Fujita et al., 2019; Ndlovu, 2014; Van Hiele, 1986; Wang & Kinzel, 2014).

In general, prospective mathematics teachers are expected to reach these levels to instruct quadrilaterals effectively and not to be among the reason why students face difficulties while learning quadrilaterals (Harper & Driskell, 2021; Ndlovu, 2014; Özdemir & Çekirdekci, 2022; Şahin & Başgül, 2020). As discussed by Avcu (2022), when teaching definitions of quadrilaterals, teachers should know that (1) the definition must include only necessary and sufficient conditions (common content knowledge) and (2) multiple equivalent and non-equivalent definitions could be generated for a quadrilateral (specialised content knowledge). This corresponds to the context of geometry education in Egypt wherein, even though middle-grade students are formally required to (1) define trapezoids, parallelograms, rhombuses, rectangles, and squares; and (2) describe relationships among them (The

Egyptian Ministry of Education and Technical Education, 2020), several local studies have reported students' inadequacy achieving these objectives (Alhannan, 2019; Almaraghy, 2020; Ekramay, 2020). Besides, a lack of research that regards teachers' side as a possible ground for such difficulties was observed.

Furthermore, and from a theoretical perspective, although several investigations explored how different groups of students define quadrilaterals (see the Introduction), a scarcity of research that utilised the results of such studies to reflect on minor levels of geometric thinking included in Van Hiele's level 3 (i.e., levels of thinking within this main level) was noticed. That is expressed by Wang and Kinzel (2014, p. 289) as follows: "According to the van Hiele model, students begin to form definitions of geometric figures and to reason deductively at Level 3. However, the model does not explicitly describe thinking at this level". Therefore, to fill this gap, the present study aims at exploring sub-levels included in Van Hiele's level 3 of geometric thinking by analysing a case of Egyptian prospective mathematics teachers' understanding of special quadrilaterals and their awareness of relationships among them.

In that sense, the current study matches Wang and Kinzel's (2014) standpoint. That is, Wang and Kinzel (2014) utilized Sfard's (2008) framework (a discursive lens) to analyse prospective teachers' geometric discourse for the case of quadrilaterals, and accordingly, they came up with particulars within Van Hiele's level 3. On the contrary, it differs from others like Çontay and Paksu (2012). In Çontay and Paksu's (2012) research, the participants were asked to classify quadrilaterals wherein their understanding of class inclusion could be clarified and ordered in terms of Van Hiele levels; as a result, each response was assigned to a specific Van Hiele level (e.g., one preservice teacher was at Van Hiele Level 3). Nonetheless, the current study (see the Methodology) hypothesized that all participants already achieved Van Hiele's levels 1 and 2, and the focus is on how they progressed towards the entire accomplishment of level 3.

Research Focus

Considering what is articulated before, the present study, on one hand, extends similar examinations (Avcu, 2022; Wang & Kinzel, 2014) and, on the other hand, clarifies the status of prospective teachers' understanding of special quadrilaterals that constitute the essential issue within the geometry content of middle grades, which is crucial to guide teacher education programs. As emphasised, further studies are needed to explore prospective middle school mathematics teachers' understanding and knowledge of quadrilaterals (Avcu, 2022; Miller, 2018). This is addressed in the current study by placing the focus on a case of prospective mathematics teachers in the Egyptian context so that some deficiencies of local research in this domain could be handled.

Research Aim and Research Questions

This study attempts to explore the progress of thinking from Van Hiele's level 2 to level 3 within the context of quadrilaterals through a case of Egyptian prospective mathematics teachers; accordingly, the following questions are addressed:

- How do prospective mathematics teachers (PMTs) define special quadrilaterals (trapezoids, parallelograms, rhombuses, rectangles, squares)?
- To what extent are PMTs aware of relationships among special quadrilaterals?
- What theoretical model would be appropriate to capture minor levels of Van Hiele's level 3 of geometric thinking within the context of quadrilaterals?

Theoretical Perspective and Prototype 1 of the Framework

In mathematics, processes of defining a concept vary between descriptive (a posteriori) and constructive (a priori) (De Villiers, 1994; De Villiers, 1998). The describing definition, the focus of this study, outlines a known object by distinguishing a few characteristic properties (Freudenthal, 1973, p. 458, as cited in De Villiers, 1998). It is executed by specifying a proper subset (the definition) of the total set of the concept properties from which all other properties could be logically deduced (De Villiers, 1998). In this study, the descriptive definition of special quadrilaterals that dominates most geometry textbooks (Usiskin et al., 2008) is sharpened, wherein the mathematics definition refers to a "collection of words which specifies all the objects to be described, and only them" (Bell & Woo, 1998, p. 56).

As argued by De Villiers (1994) on the relationship between defining and classifying, which are related to each other, the mathematical concept could be defined within either a hierarchical classification or a partitional one. While hierarchical classification represents the concept within a subset of more general concepts, partitional classification identifies each concept as disjoint from another (Fujita & Jones, 2007; Ulger & Broutin, 2017). For instance, the trapezoid could be defined as a quadrilateral with (1) at least one pair of opposite sides parallel or (2) only one pair of opposite sides parallel; that is, the former is hierarchical since it recognized parallelograms as trapezoids, and the latter is partitional wherein parallelograms were excluded from trapezoids. Both definitions are mathematically correct; however, in most cases, the hierarchical definition stays more economical (specify only necessary and sufficient conditions) than the partitional one that requires adding more properties (more than necessary characteristics) to exclude other related concepts (De Villiers, 1994; Fujita et al., 2019). Hence, the hierarchical definition is favoured in mathematics school curricula (Fujita & Jones, 2007) since it reflects a further understanding of the relationships among quadrilaterals (inclusion relationships) and helps in advancing deductive reasoning and proof (Erez & Yerushalmy, 2006, as cited in Ulger & Broutin, 2017; Fujita, 2012).

The above-documented classification of De Villiers (1994) was prescribed by Usiskin et al. (2008) using terms exclusive and inclusive that match the partitional and hierarchical definitions, respectively. While the exclusive definition is correct for a specific quadrilateral, the inclusive one would be valid for a family of quadrilaterals. Recently, Bütüner and Filiz (2017) classified Turkish teachers' definitions of quadrilaterals as erroneous, prototype, and hierarchical. Although both determine proper definitions, the former is correct for a specific concept only, while the latter stands valid for this concept and its special cases (group). Again, prototype and hierarchical definitions correspond to De Villiers's (1994) classification of partitional and hierarchical, respectively.

Based on this argument, it was hypothesized that PMTs' special quadrilaterals understanding could be ordered as follows: (L0) faulty, (L1) partitional-uneconomical, (L2) partitional-economical, (L3) hierarchical-uneconomical, and (L4) hierarchical-economical that reflect the entire attainment of Van Hiele's level 3 in the context of quadrilaterals (see Table 1). Noting that economical and uneconomical correspond to Zazkis and Leikin's (2008) notions of minimal and non-minimal definitions, respectively; while the former includes parts that cannot be implied from other properties of the definition, the latter contains additional properties more than the necessary ones. Besides, the concept definition, according to this study, refers to the *personal concept definition* that is defined by Tall and Vinner (1981, p. 152) as "the form of words that the student uses for his own explanation of his (evoked) concept image", which might or might not, equal to the formal concept definition.

Table 1

Levels of Special Quadrilaterals Understanding "Prototype 1"

Levels of a quadrilateral understanding		*Characteristics of the level*	
		Attributes of a quadrilateral	*Consideration of Inclusion relationships*
L2. Faulty (≤ Van Hiele's level 2)		X (Not relevant; not sufficient; not necessary)	X (Not considered)
Level 2 < Partial understanding < level 3	*L2.1 Partitional uneconomical*	√ (More than the sufficient attributes)	X (Not considered)
	L2.2 Partitional economical	√ (Only the sufficient attributes)	X (Not considered)
	L2.3 Hierarchical uneconomical	√ (More than the sufficient attributes)	√ (Considered)
Full understanding = level 3	*L3. Hierarchical economical*	√ (Only the sufficient attributes)	√ (Considered)

Table 1 describes the predetermined analytical framework inspired by similar previous studies (Avcu, 2022; Ndlovu, 2014). In detail, Ndlovu (2014) classified understanding of definitions into non-standard, economical, uneconomical, very uneconomical, incorrect, and uninterpretable. Furthermore, Ndlovu's (2014, p. 6650) study results concluded that "allowances should be made for differences in geometrical understandings within individual students and that the Van Hiele theory be interpreted in broader terms to allow for partial attainment of a level", which confirms the viewpoint of this study. Alike, based on the sufficiency and necessity of articulated properties of definitions, Avcu's (2022) framework (adapted from Zazkis and Leikin (2008)) scored degrees of correctness of definitions into 0, 1, and 2 points to indicate neither necessary nor sufficient conditions (0), necessary but not sufficient conditions or sufficient but not necessary conditions (1), and necessary and sufficient conditions (2), respectively. Moreover, the correct definitions with necessary and sufficient conditions were further classified as minimal, barely not minimal, or significantly not minimal (or improper terms).

Research Methodology

General Background

The present study represents a case study that is viewed by Creswell (2007, p. 73) as "a methodology, a type of design in qualitative research, or an object of study, as well as a product of the inquiry". Specifically, it defines a single instrumental case study that fits the study perspective; that is, the concern of Van Hiele's level 3 of geometric thinking was focused; accordingly, a bounded case of PMTs was selected to illustrate this concern (Creswell, 2007). This bounded case – the study participants – already accomplished Van Hiele's level 2, as detailed in the next section.

Participants

The data were collected from a purposeful sample of PMTs (24 participants) who accomplished Van Hiele's level 2. This intentional selection of participants happened in light of the standard of mastering level 2 of geometric thinking; thus, the intended progression of thinking toward level 3 could be sharpened. All the study participants were regular graduate students enrolled in a diploma program that aims to prepare non-educational graduates to be mathematics teachers during the two consecutive academic years 2021/2022 and 2022/2023.

Generally, those participants, ranging between 25-40-year-old, shared similar common knowledge of mathematics since they all graduated from technical faculties (science and engineering) and joined this diploma to be able to work in the education field. In addition, they gained some experience on the topic of special quadrilaterals during the Teaching Methods course that was mandatory to register.

Procedures of Data Collection and Analysis

At first and before collecting the primary data, it was essential to ensure that the participants had already reached Van Hiele's level 2 where they could list the properties of special quadrilaterals (Ndlovu, 2014). To do so, during the microteaching course, a group of PMTs was invited to dram instances of a trapezoid, parallelogram, rhombus, rectangle, and square; and to determine each quadrilateral property. Accordingly, the study sample was selected; it specified 24 participants who could distinguish quadrilaterals by related properties (i.e., meet the study criteria).

Next, considering multiple relevant studies that investigated learners' knowledge of quadrilaterals and their understanding of the inclusion relationships (Avcu, 2022; Bütüner & Filiz, 2017; Erdogan & Dur, 2014; Fujita & Jones, 2007; Ndlovu, 2014; Syamsuddin, 2019), a survey was used to collecting the principal data required for this study. This was done after obtaining the demanded permissions from the faculty and cordial consensus from the participants to share their responses anonymously merely for research concerns; alternatively, their answers would not be considered for the formal assessment of the course.

This survey consisted of two main questions: (Q1) *Define the following special quadrilaterals (in one or more different ways)*: trapezoid, parallelogram, rhombus, rectangle, and square; (Q2) *Represent the relationship among the quadrilaterals defined in Q1*. The participants were asked to respond to these questions in their language (The Arabic language) with no time rules; however, it took them around 45 minutes to complete it. Although the given questions were consistently found in the literature, they were further judged by local mathematics educators regarding their language clarity and suitability to explore PMTs' understanding of special quadrilaterals.

The participants' answers were collected and analysed within two cycles. The first was accomplished in November 2021, wherein 16 PMTs' responses to both questions were scrutinized considering Prototype 1. In other words, the researchers evaluated the possibility of each response being designated to each level of understanding described in Prototype 1. One year later, in November 2022, 8 PMTs partook in this study by answering the same questions, in which their responses helped validate Prototype 1. That is, the researchers replicated the same analytical process performed in the first cycle to confirm whether all responses could be included in the suggested levels. As shown in Tables 3-7 (see the Results), definitions written in black, blue, and red colours denote PMTs' responses that emerged in the first cycle, second cycle, and both cycles, respectively.

Besides the researchers' agreement on the classification of responses, and to ensure the validity of the content analysis in this study, it was compared to categorizations of data in other related studies like Duatepe-Paksu et al. (2012) and Bütüner and Filiz (2017). For example, in Bütüner and Filiz's (2017) analysis, mathematics teachers' definitions of special quadrilaterals were categorized as hierarchical, prototype, and erroneous or partially hierarchical. Bütüner and Filiz (2017) first determined the validity of a definition; if it was not valid, it was classified as erroneous/ partially hierarchical/ partially accurate, which matches the level of faulty in the current study. On the other side, if it was correct, they classified it as either hierarchical or prototype; hierarchical if it was proper for all special cases of the given quadrilateral, while if not, it should be a prototype definition. This seems as if the prototype definition in Bütüner and Filiz's (2017) study combines the partitional levels (economical/ uneconomical) proposed in the current study. Concretely, in Bütüner and Filiz's (2017) analysis, they noted that a trapezoid could be defined (a) hierarchically as a quadrilateral having two at least a pair of parallel sides or (b) prototypically as a quadrilateral having only one pair of parallel sides; that corresponds to the hierarchical economical and partitional economical definitions of a trapezoid, respectively (see Table 3).

In addition to the PMTs' written responses to Q1 in the survey (the data required to answer the first research question), 24 representations of the relationship among special quadrilaterals were portrayed as answers to Q2. Since the main objective of this question was to explore the participants' awareness of the relationships among quadrilaterals, clinical interviews were conducted with the participants to confirm the links that appeared in their represented models. Using both sources of data (written responses and interviews) helped ensure the consistency and validity of the results. For example, the 3 participants who drew Model A (see Figure 6 and the Appendix) declared that there is no relationship between trapezoids and parallelograms. Yet, rhombuses, rectangles, and squares are all special cases of parallelograms. Also, squares maintain the properties of both rhombuses and rectangles. Accordingly, the researchers judged that those participants were aware of all links expressed in Figure 9, except the four associations that include trapezoids. Similarly, after interviewing the participants who displayed relationships as Model B (see Figure 7), their understanding of parallelogram-rhombus, parallelogram-rectangle, and parallelogram-square linkages was verified.

Research Results and Discussion

To answer the first research question (How do PMTs define special quadrilaterals?), the participants were encouraged to document one (or two) definitions of a trapezoid, parallelogram, rhombus, rectangle, and square; still, some participants offered a single definition for each (see Tables 3-7).

Not only participants' written responses were analysed, but also the figures drawn by some supported interpret these responses more confidently. As a result (see Figure 2), around 70% of the delivered definitions were correct, while the incorrect ones had common themes of including (a) necessary but not sufficient conditions and (b) inappropriate terms. The description of these incorrect definitions categories is out of the current study focus; besides, it has already been detailed in Avcu's (2022) investigation.

This high percentage of correct responses was expected since the selected sample already attained level 2 of geometric thinking (see the Methodology). More expressly, the participants demonstrated remarkable performance in generating correct definitions of rhombuses and squares compared to the case of trapezoids and rectangles, which match studies like Miller (2018) and Türnüklü et al. (2013). Furthermore, this result remains close to the results of Ndlovu (2014), who declared that more than 68% of the pre-service teachers (16 participants) were able to define trapezoids (75%), parallelograms (75%), rhombuses (68.6%), rectangles (87.5%), and squares (93.8%) correctly. Also, Erdogan and Dur (2014) reported that the percentages of pre-service teachers' correct definitions of trapezoids, parallelograms, rhombuses, rectangles, and squares were 46%, 96%, 33%, 96%, and 95%, respectively. On the contrary, percentages of correct definitions of special quadrilaterals provided by pre-service teachers were low in other studies such as Duatepe-Paksu et al. (2012), Güner and Gülten (2016), and Özdemir and Çekirdekci (2022). Yet, the correct definitions altered in terms of being economical and inclusive, as determined in prototype 1.

Figure 2

Percentages of Faulty vs. Correct Definitions of Special Quadrilaterals

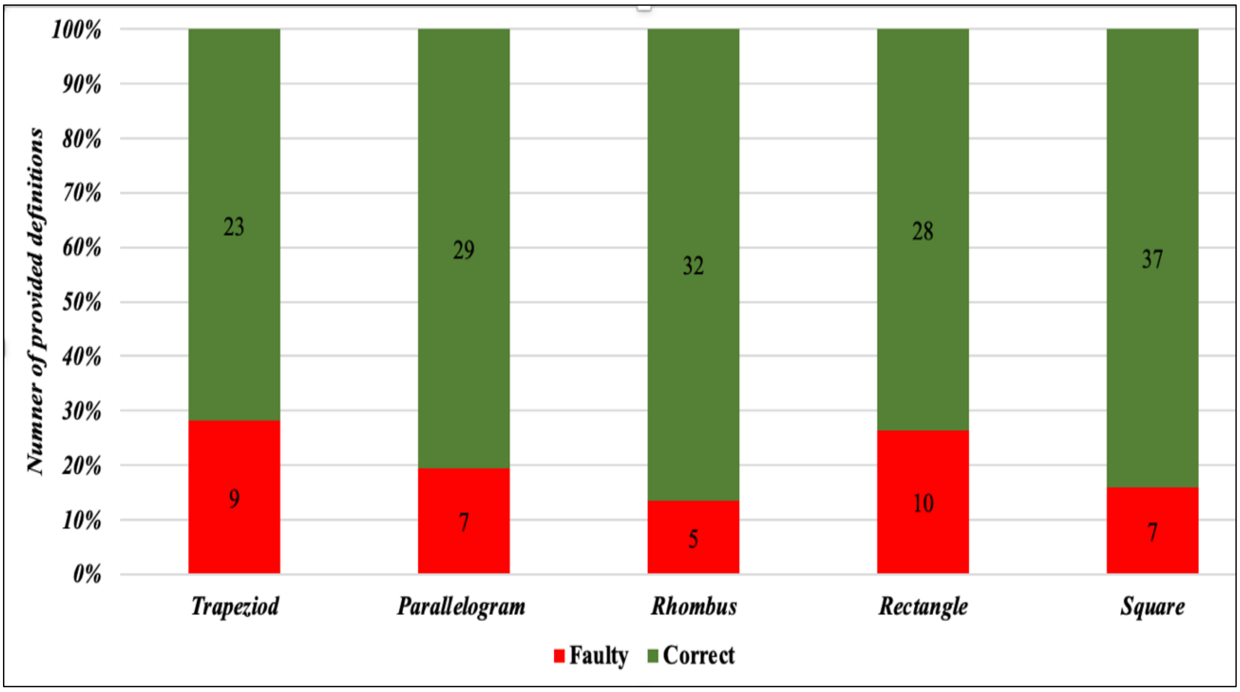


Table 3

PMTs' Levels of Trapezoid Definition Understanding

Levels	*PMTs' personal definitions of trapezoid*	*No.*
L2. Faulty	• A quadrilateral with four unequal and non-parallel sides. • A parallelogram with only two opposite parallel/ non-parallel sides. (4) • A parallelogram whose opposite sides are not equal. *Necessary but not sufficient conditions:* • A quadrilateral with four unequal sides. • A quadrilateral in which each acute angle is opposite to an obtuse angle. *Inappropriate terms:* • A closed geometric figure with only two sides parallel.	***9***
L2.1 Partitional uneconomical	• A polygon whose sum of interior angles equals 360 degrees with only a pair of opposite sides parallel.	***1***
L2.2 Partitional economical	• A quadrilateral with only one pair of opposite sides parallel. (13) • A quadrilateral with only one pair of parallel unequal sides. (4)	***17***
L2.3 Hierarchical uneconomical		***0***
L3 Hierarchical economical	• A quadrilateral with a pair of opposite sides parallel. (5)	***5***
	Total number of provided definitions of trapezoids	***32***

Table 4

PMTs' levels of parallelogram definition understanding

Levels	PMTs' personal definitions of parallelogram	No.
L2. Faulty	• A quadrilateral whose diagonals are equal and perpendicular bisectors of each other. • A regular quadrilateral in which each two opposite sides parallel. • A rectangle with two acute angles. *Necessary but not sufficient* • A quadrilateral with a pair of parallel side. • A quadrilateral with a pair of parallel sides and two adjacent complementary angles. *Inappropriate terms* • A polygon in which each two opposite sides are parallel and equal in length. • A close geometric figure with two opposite sides parallel and equal in length.	7
L2.1 Partitional uneconomical		0
L2.2 Partitional economical	• A quadrilateral with two unequal adjacent sides and two pairs of opposite sides equal. • A quadrilateral whose unequal non-perpendicular diagonals, but they bisect each other.	2
L2.3 Hierarchical uneconomical	• A polygon whose sum of the interior angles is 360 degrees, and two pairs of opposite sides parallel. • A quadrilateral with two pairs of opposite sides parallel and equal in length. (9) • A quadrilateral with two pairs of opposite sides parallel and equal in length, and whose diagonals bisect each other.	11
L3 Hierarchical economical	• A quadrilateral with two pairs of opposite sides parallel. (13) • A quadrilateral with two pairs of opposite sides equal. (2) • A quadrilateral whose diagonals bisect each other.	16
	Total number of provided definitions of parallelograms	36

Table 5

PMTs' Levels of Rhombus Definition Understanding

Levels	PMTs' personal definitions of rhombus	No
L2. Faulty	• A quadrilateral whose sides are equal in length and whose angles are right angles. • A square with two acute angles. *Necessary but not sufficient* • A quadrilateral with two pairs of opposite sides parallel and any two adjacent angles are supplementary. • A quadrilateral with two adjacent sides equal and perpendicular diagonals. • A quadrilateral whose diagonals are perpendicular but not equal.	5
L2.1 Partitional uneconomical	• A quadrilateral with equal sides and unequal perpendicular diagonals. (2)	2
L2.2 Partitional economical	• A quadrilateral with equal sides and no right angles.	1
L2.3 Hierarchical uneconomical	• A quadrilateral with four sides equal and perpendicular diagonals. • A quadrilateral with four sides and four angles, two pairs of opposite sides parallel, and its diagonals bisect each other at 90 degrees. • A quadrilateral with four sides equal and two pairs of opposite sides parallel. • A parallelogram in which each two adjacent sides are equal in length (3).	8

	• A parallelogram whose diagonals are perpendicular and bisect each other. • A parallelogram whose sides are equal, and diagonals are perpendicular to each other.	
L3 Hierarchical economical	• A quadrilateral whose sides are equal. (5) • A parallelogram with two adjacent sides equal. (11) • A parallelogram with perpendicular diagonals. (5)	***21***
	Total number of provided definitions of rhombuses	***37***

Table 6

PMTs' Levels of Rectangle Definition Understanding

Levels	***PMTs' personal definitions of rectangle***	***No.***
L2. Faulty	• A quadrilateral which could be divided into two squares. • A square whose sides are equal in length and whose diagonals are perpendicular. *Necessary but not sufficient* • A quadrilateral with two pairs of opposite equal sides. (3) • A quadrilateral with two adjacent perpendicular unequal sides. • A parallelogram with two pairs of opposite parallel and equal sides. (2) *Inappropriate terms* • A geometric figure in which each two opposite sides are parallel and equal in length. • A geometric figure with only one right angle.	**10**
L2.1 Partitional uneconomical		**0**
L2.2 Partitional economical		**0**
L2.3 Hierarchical uneconomical	• A quadrilateral with two pairs of opposite sides parallel and one right angle (2). • A quadrilateral with two pairs of opposite sides parallel and equal in length; and whose all angles are right. (2) • A quadrilateral with two pairs of opposite sides equal and one right angle. • A quadrilateral with two pairs of opposite sides equal and at least one right angle. • A quadrilateral with two pairs of opposite sides parallel and four right angles. • A parallelogram with four right angles and equal diagonals. (3)	**10**
L3 Hierarchical economical	• A parallelogram with one right angle. (15) • A parallelogram whose diagonals are equal in length. (3)	**18**
	Total number of provided definitions of rectangles	**38**

Table 7

PMTs' Levels of Square Definition Understanding

Levels	***PMTs' personal definitions of square***	***No.***
L2. Faulty	*Necessary but not sufficient* • A quadrilateral with two adjacent equal perpendicular sides. (3) • A quadrilateral whose all sides are equal in length.	**7**

	• A parallelogram with two adjacent sides equal and whose diagonals bisects each other. • A parallelogram whose diagonals are equal in length. • A four-sided geometric figure with two pairs of opposite sides parallel and all sides are equal in length.	
L2.1 Partitional uneconomical	• A quadrilateral with two pairs of opposite sides parallel, all sides are equal, and all angles are right. • A quadrilateral with equal sides, equal angles, and perpendicular diagonals. • A quadrilateral whose sides are equal, and all angles are right angles. (3) • A regular quadrilateral with one right angle and two adjacent sides equal. • A quadrilateral whose diagonals are equal, bisect each other, perpendicular, and angles bisectors.	**7**
L2.2 Partitional economical	• A quadrilateral with all sides equal and one right angle. (3)	**3**
L2.3 Hierarchical uneconomical	• A rhombus whose diagonals are equal, perpendicular, and bisect each other. • A parallelogram whose sides are equal, and all angles are right angles. (3) • A parallelogram whose sides are equal, angles are right angles, and diagonals are equal and perpendicular to each other. • A parallelogram with equal sides, angles are right angles, and two pairs of opposite sides are parallel.	**6**
L3 Hierarchical economical	• A parallelogram with two adjacent sides equal and one right angle. (11) • A parallelogram whose diagonals are equal and perpendicular to each other. • A rhombus with one right angle. (4) • A rectangle with two adjacent sides equal. (4) • A rectangle whose diagonals are perpendicular.	**21**
	Total number of provided definitions of squares	**44**

Based on prototype 1, the data in the above tables are represented in Figure 3. It shows that the two levels of hierarchical economical and hierarchical uneconomical were evident (around 54% and 24%, respectively) compared to both partitional economical and partitional uneconomical (around 15% and 7%, respectively); more precisely, the partitional uneconomical level seemed absent in the participants' definitions, except for some cases of squares, rhombuses, and trapezoids.

In that sense, the researchers anticipated that the determinants of Prototype 1 might be overlapped. Admitting the description of hierarchical and partitional definitions in Erdogan and Dur's (2014, p. 111) study, "The definitions that contain sufficient information to exclude non-examples are called partitional definitions. The definitions that contain all objects including all of the properties and that are more economical and shorter than partitional definitions are called hierarchical definitions", indicates a relationship between hierarchies and the economics of the definition. This was initially noted by De Villiers (1994, p. 15) as one function of the hierarchical classification; it "leads to more economical definitions of concepts"; thus, the "hierarchical definition is shorter than a partitional one which has to include additional properties". Alternatively stated, defining quadrilaterals hierarchically promotes constructing the definition more economically. This is noticeable in Figure 3 when comparing the (a) hierarchical economical (54%) to the partitional economical (15%); and (b) the hierarchical-economical to the hierarchical uneconomical (24%). Therefore, and considering this argument, the researchers decided to separate these levels and rearrange the data considering one property either by focusing on the economy or inclusion, as displayed in Figures 4 and 5.

Figure 3

Prototype 1 Levels of PMTs' Definition Understanding

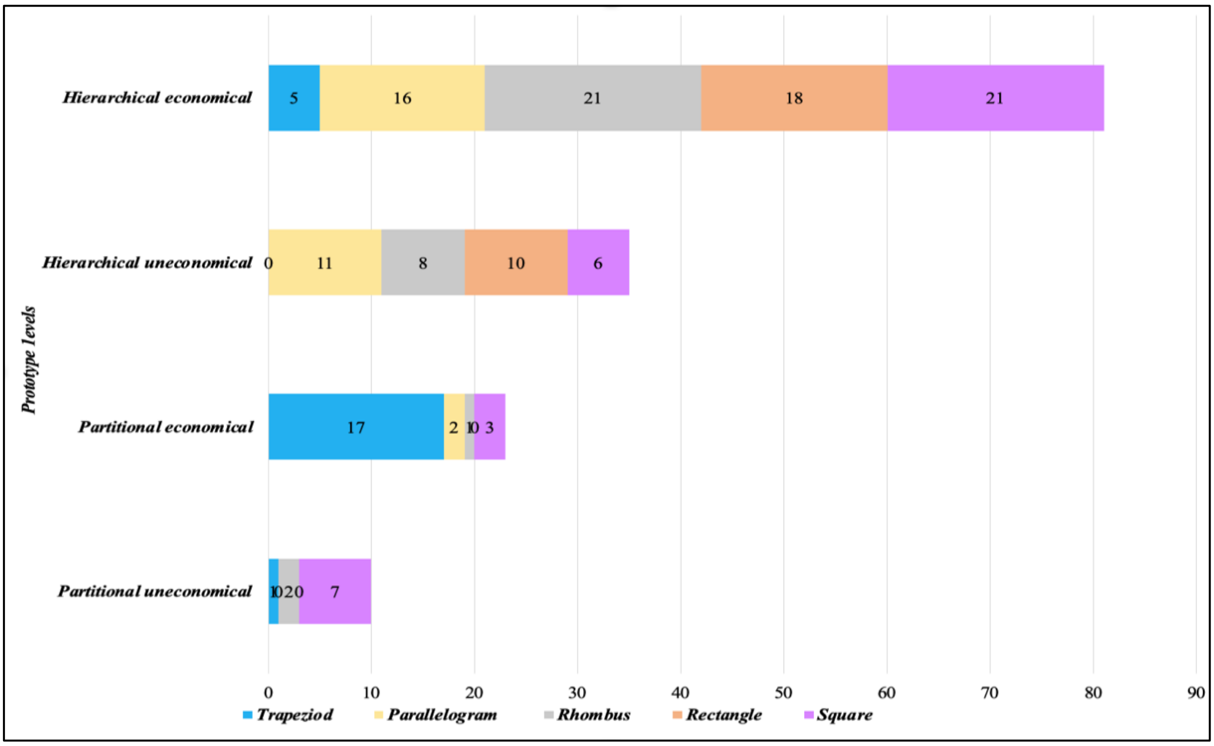


Figure 4

Levels of Definition Understanding by Inclusion

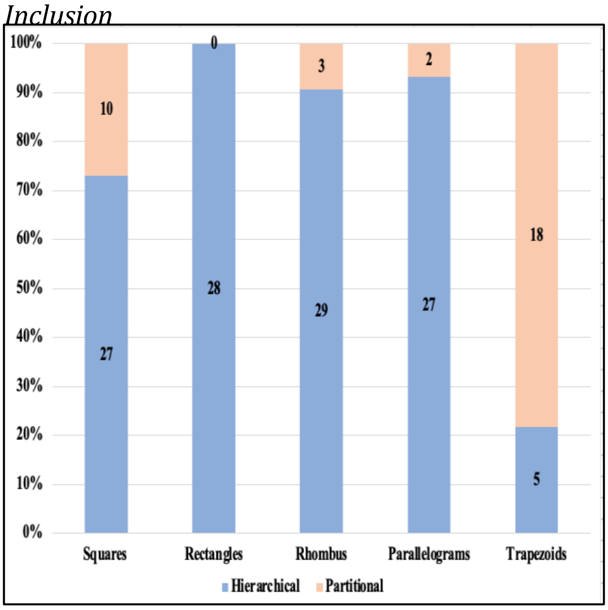
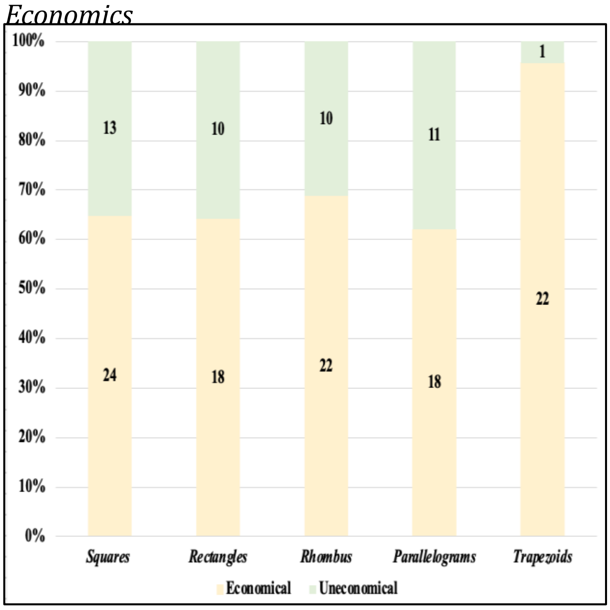


Figure 5

Levels of Definition Understanding by Economics



First, looking to Figure 4 conveys a result that disagrees with multiple previous studies. According to these studies, most learners in various grades face difficulty in defining special quadrilaterals because of their lack of understanding of the hierarchical classification and inclusion relations (Fujita & Jones, 2007; Fujita, 2012; Jones et al., 2002; Pickreign, 2007; Ulger & Broutin, 2017), while, except for the case of trapezoids, more than 70% of participants of this study already defined the special quadrilaterals hierarchically. Most probable, this occurred because of the characteristics of the study sample, who already accomplished Van Hiele's level 2, besides their awareness of the relationships among these quadrilaterals since the Egyptian school textbook of grade 7 provides a clear representation of these relationships and recommends teachers to highlight it. That is underlined in Žilková's (2015) study, wherein country context, level of education, cognitive level of learners, and textbooks' views are possible determinants when explaining the students' definition of quadrilaterals.

Considering the above-documented finding, the researchers anticipated the reason why the participants could, hierarchically, define the special quadrilaterals is the participants' awareness of relationships among these quadrilaterals. In other words, the researchers believed that the participants reflected on their internal representations (mental models) of relationships among special quadrilaterals to help them while defining. To ensure that, the participants were requested to portray such relationships (Q2). Accordingly, their representations were classified into three models (see Figures 6, 7, and 8; also, authentic instances are provided in the appendix); 3, 3, and 18 representations for models A, B, and C, respectively.

Moreover, the 24 participants' drawings were analysed by the reciprocal links between every two quadrilaterals to answer the second research question. After verifying the participants' understanding of relationships through brief interviews (see the Methodology), it was found that most PMTs were conscious of relationships among quadrilaterals, except for trapezoids, as shown in Figure 9. They precisely comprehended that (a) rhombuses, rectangles, and squares are special parallelograms, also

(b) rhombuses and rectangles could be transformed into squares when adding a property. Still, the present study participants' awareness of the relationships among quadrilaterals is high compared to other studies like Okazaki and Fujita (2007) and Žilková (2015).

In detail, although 93%, 72%, 41%, and 25% of Erdogan and Dur's (2014), Žilková's (2015), Okazaki and Fujita's (2007), and Ndlovu's (2014) study participants, respectively could identify the Parallelogram- Rhombus relationship, all the current study participants did so. Similarly, percentages of recognising the Parallelogram- Rectangle relationship were 93% (Erdogan & Dur, 2014), 44% (Žilková, 2015), 40% (Okazaki & Fujita, 2007), and 31% (Ndlovu, 2014) compared to 100% in the present study. Again, for the Rhombus-Square relationship, it was acknowledged by 87.5% (21 cases) of this study participants compared to 63% (Ndlovu, 2014), 25% (Okazaki & Fujita, 2007), and 15% (Žilková, 2015).

Figure 6

Model A Representation of Relationships Among Special Quadrilaterals

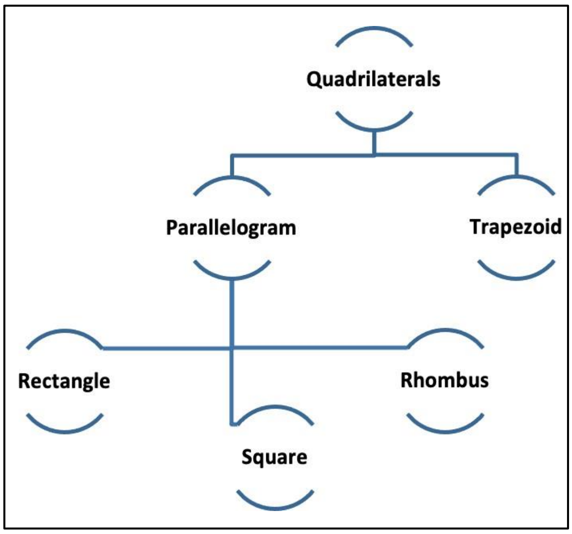


Figure 7

Model B Representation of Relationships Among Special Quadrilaterals

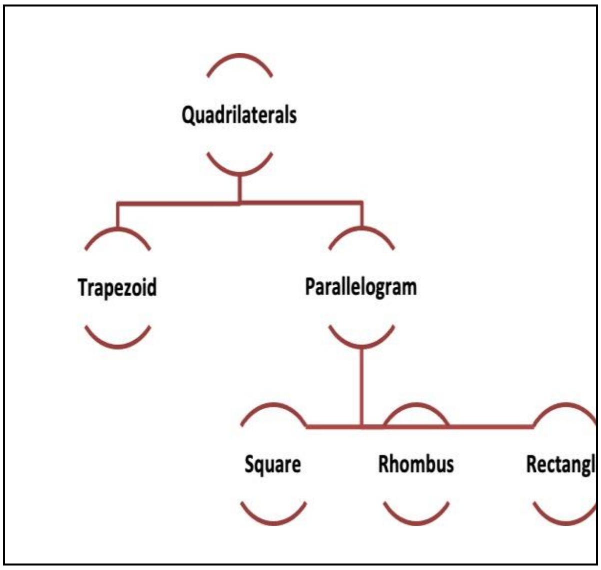


Figure 8

Model C Representation of Relationships Among Special Quadrilaterals

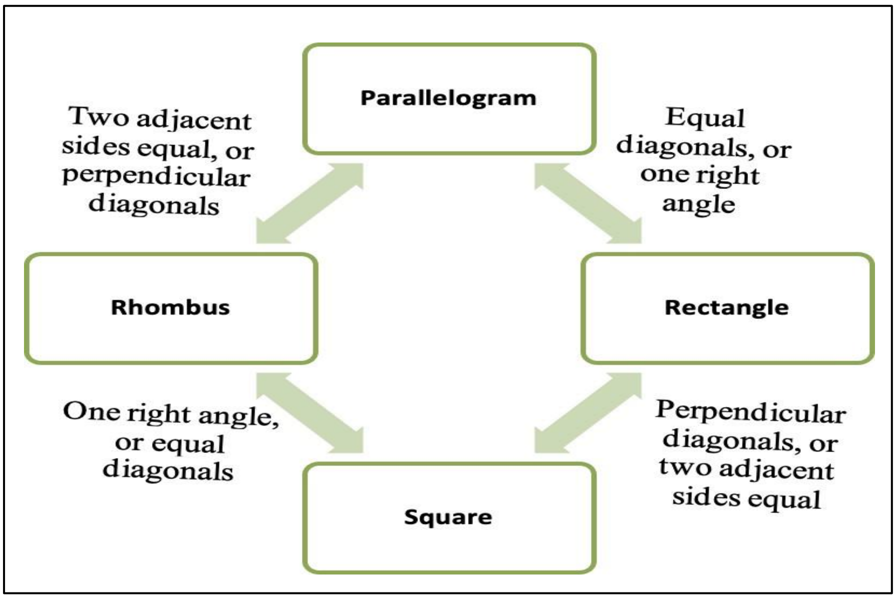
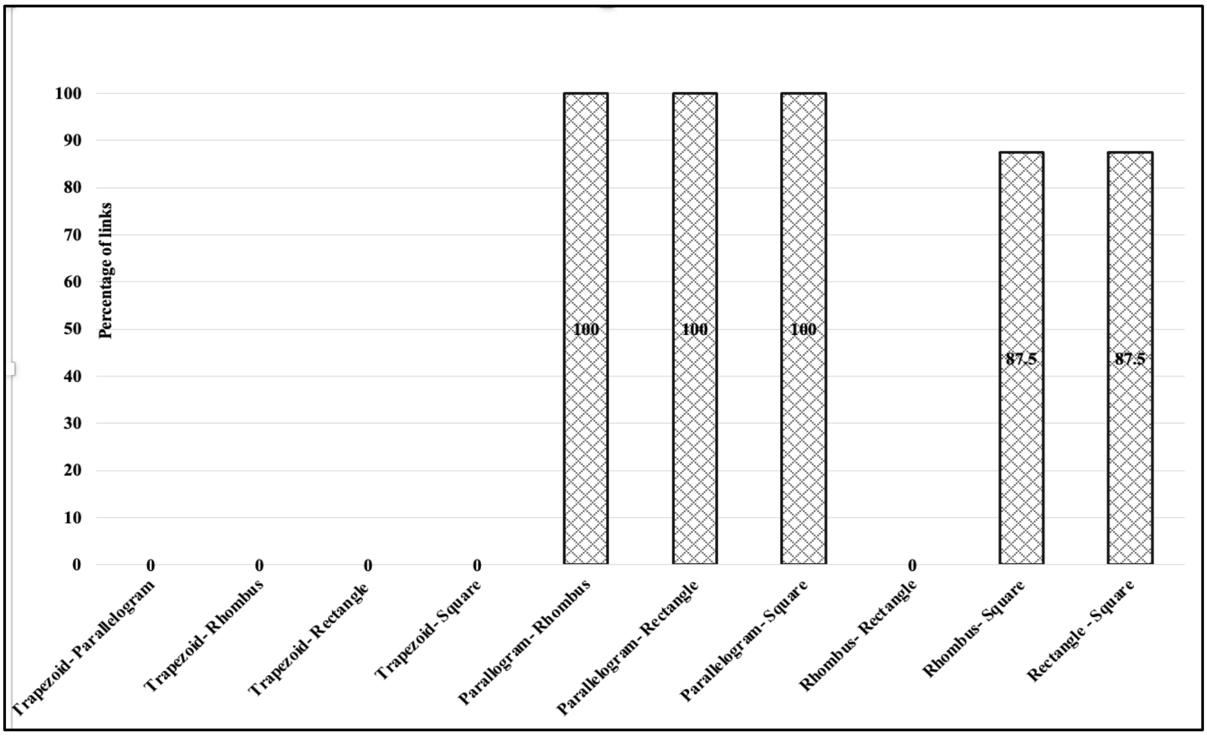


Figure 9

PMTs' Awareness of Relationships Among Special Quadrilaterals



Second, exploring Figure 5 shows that more than 60% of the provided correct definitions are economical, which stays high compared to other studies like Ndlovu (2014) and Avcu (2022). In detail, the percentages of correct economical were about 44%, 6%, 6%, 13%, and 19% in Ndlovu's (2014) study corresponded to 87%, 45%, 36%, 1%, and 3% in Avcu's (2022) research for trapezoids, parallelograms, rhombuses, rectangles, and squares, respectively.

Third, comparing the depicted data in Figures 4 and 5 reveals that grading the definition understanding in terms of the economy appears more suitable than focusing on the inclusion relationships; if and only if the learners did accomplish Van Hiele's level 2 of geometric thinking and they are progressing towards level 3. As exhibited in both figures, percentages of PMTs' identification of definitions hierarchically constituted 93%, 91%, 100%, and 73% compared to 62%, 69%, 64%, and 65% for parallelograms, rhombuses, rectangles, and squares, respectively, if these definitions were classified based on the economy property. That is, the range of the first distribution (27%) is higher than the second (7%), which indicates that categorising definitions economically minimizes the variability that might appear among individuals' understanding of special quadrilaterals.

Accordingly, it can be argued that if concentrating on the economics of a definition, it is possible, consequently, to reach the inclusion relationships. For example, instead of defining a square as *a quadrilateral with equal sides, equal angles, and perpendicular diagonals- partitional uneconomical* (see Table 7), we would change it to *a rhombus with equal angles and perpendicular diagonals- hierarchical uneconomical*; hence, to be more economical, it would be *a rhombus with equal angles- hierarchical economical* because the concept "rhombus" does possess the perpendicular diagonals. In that sense, the definition of the square has been advanced from the level of *partitional uneconomical* to the *hierarchical economical*.

Such an argument matches De Villiers (1986), who demonstrated that the "additional unnecessary and untrue attributes [in learners' definitions] are the main causes of difficulties in the understanding of inclusion relations for quadrilaterals" (Ndlovu, 2014, p. 6649). Thus, excluding these unnecessary attributes to construct the definition economically would influence the comprehension of the inclusion relationships. Recently, for the importance of definition economics, Kabaca (2017) has recommended operating the three properties of diagonals (i.e., congruent, perpendicular, dividing each other in ratio) or any possible combinations of these properties to define quadrilateral, which would be more economically than utilizing properties of angles and sides.

This probably explains why Zazkis and Leikin's (2008) and Avcu (2022), who followed them, concentrated on the extent of minimality (minimal, barely not minimal, or -not minimal) while categorizing prospective secondary school teachers' correct appropriate definitions of a square. Also, Ndlovu (2014) classified pre-service teachers' understanding of geometrical definitions and class inclusion as non-standard (correct economical not usually provided in textbooks), correct economical, correct uneconomical, correct very uneconomical, and uninterpretable. It also strengthened the economics of the correct definitions, regardless of being partitional or hierarchical.

Additionally, it is crucial to note that the answer to Q2 that expresses PMTs' awareness of the inclusion relationships (see Figure 9) is consistent with the answer to Q1. Concretely, looking to Figure 4 displays that about 22%, 93%, 91%, 100%, and 73% PMTs could define trapezoids, parallelograms, rhombuses, rectangles, and squares, respectively hierarchically, which also admits PMTs' recognition of the inclusion relationships (except for trapezoids). This finding suggests that prioritizing the economics of definitions should be complemented by admitting learners' awareness of the direction/extent of inclusion. That is, leveling learners' understanding of special quadrilaterals has to regard the amalgam of economics and learners' awareness -represented by their intentions- (Prototype 2) instead of economics and hierarchies (Prototype 1). Acknowledging this would solve Ndlovu's (2014, p. 6650) dilemma of which "students can be at different levels of van Hiele understanding for different aspects/concepts of a geometrical topic".

Based on this argument, Prototype 1 could be further revised to focus essentially on the economics of the definitions; yet, learners' awareness of relationships among these definitions should be considered (i.e., whether the learner tries, consciously, to exclude/include the defined concept from the set of general concepts). Therefore, the resultant theoretical model (see Figure 8) represents the answer to the third research question; it describes the minor levels of Van Hiele's level 3 of geometric thinking within the context of quadrilaterals.

Table 8

Levels of Special Quadrilaterals Understanding "Prototype 2"

	Characteristics the concept definition in this level	***Examples from PMTs' responses***
Faulty *≤Van Hiele's level 2*	• contains necessary but insufficient conditions. • includes inappropriate terms, wherein there is a possibility for maintaining some non-examples of the concept. • begins with a special concept to define a more general one.	The parallelogram is a quadrilateral with a pair of parallel sides. The trapezoid is a closed geometric figure with only two sides parallel. [This could be a hexagon] The parallelogram is a rectangle with two acute angles.
Slightly economical (Moves towards the direction of inclusion)	• maintains a whole statement (definition) that could be replaced by a specific concept (term). • holds more than sufficient conditions (additional properties more than necessary ones).	The rhombus is a quadrilateral with four sides equal and perpendicular diagonals. *[The definition (a) excludes rhombuses from parallelograms; however, it recognizes squares as a*

Van Hiele's level 2< Partial understanding < level 3	• contains some properties that could be deduced from another. • intends to verify the concept definition by adding supplementary properties (i.e., a list of redundant properties). • indicates either the general set of concepts where the defined concept belongs or the special cases from the defined concept, or both (recognises subsets and supersets of the defined concept).	*subset of rhombuses. (b) contains a property of perpendicular diagonals that could be deduced from having equal sides]*
Fairly economical (Moves towards the direction of partitional) *Van Hiele's level 2< Partial understanding < level 3*	• holds more than sufficient conditions. • contains extra properties that aim to specify the concept from other related concepts. • excludes the defined concept from the more general one; also excludes some/all special cases from this defined concept.	The rhombus is a quadrilateral with equal sides and no right angles. *[The definition (a) excludes (specifies) rhombuses from both parallelograms and squares. (b) The property of "no right angles" is not necessary, yet, it is added to confine the concept.]*
Economical *Full understanding = Van Hiele's level 3*	• contains the sufficient conditions only (no words could be removed). • no property could be deduced from another (no superfluous information). • admit the defined concept within a set of more generalised concepts; also, recognizes the set of special concepts from this defined concept.	A rhombus is a parallelogram with two adjacent sides equal. *[The definition admits rhombuses to be special parallelograms; also, squares as included within the set of rhombuses.]*

Table 8 (Prototype 2) answers the third research question; it concretises the minor levels of Van Hiele's level 3 of geometric thinking within the context of quadrilaterals. These levels that are proposed in Prototype 2 appear consistent with the framework employed in the recent study of Pagiling and Nur'aini (2022) where the levels of prototypical, partial prototypical, and hierarchical were used to analyse secondary mathematics teachers' special content knowledge of defining and classifying quadrilaterals. Furthermore, it responds to Ndlovu's (2014, p. 6650) proposal on adapting Van Hiele's levels where "Van Hiele theory be interpreted in broader terms to allow for partial attainment of a level". Alternatively stated, the difficulty of categorising definition understanding considering the concept definition and the inclusion relationships together, since it is probable for a learner to be at a level (or in transition) when defining a concept and at another level when determining class inclusion, could be overcome through the proposed framework (Prototype 2). It essentially strengthens the extent of the economics of the concept definition, which includes the specified words (precision in terminologies) and the intention of utilising these words (towards partitional or inclusion). Hence, the individual's level of geometric thinking could be determined based on the consistency between his personal concept definition and awareness of the inclusion/exclusion of other related concepts.

Conclusions and Implications

In terms of the importance of processes of defining and classifying in learning and teaching geometric thinking, this study aimed to track the progress of thinking from Van Hiele's level 2 to level 3 based on teachers' definitions of special quadrilaterals and their awareness of the relationships among them. To do so, a bounded purposeful case of prospective mathematics teachers was selected and requested to (a) define these special quadrilaterals: trapezoids, parallelograms, rhombuses, rectangles, and squares; and (b) represent the relationships among these quadrilaterals. Hence, their responses were analysed and assigned to Prototpe1 levels. These levels were developed theoretically as Faulty, Partitional uneconomical, Partitional economical, Hierarchical uneconomical, and Hierarchical economical.

Analysing the participants' answers allowed developing Prototype 1 to Prototype 2, which reconceptualised these levels into faulty, slightly economical, fairly economical, and economical. Prototype 2 ranks learners' understanding of special quadrilaterals according to (a) the economics of the concept definition and (b) their awareness of relationships among other related definitions to this concept, expressed by their intention to exclude or include the concept within another set of concepts.

One critical point concerning the developed framework is its acknowledgment of De Villiers's (1994) remark about emphasising the partitioned definition as equivalent to the hierarchical; "a partition classification is equally acceptable and a frequently employed method in mathematics" (De Villiers, 1994, p. 17). That is, this framework highlights the economics of the definition relatively than strengthening being hierarchical or partitional. Yet, learner's awareness/ intention towards excluding/including the defined concept from its general set have to be regarded.

The study also revealed the effectiveness of the textbook as a vital determinant of learners' definition of quadrilaterals, as previously remarked in investigations like Duatepe-Paksu et al. (2012) and Žilková (2015). Concretely, it has been interpreted that participants' adequate level of awareness of the inclusion relationship among special quadrilaterals, which was consistent with their personal concept definitions, might be caused by their experience with the school textbooks of middle grades, specifically the grade 7 textbook that gives a visual representation of these relationships. Nonetheless, simply appointing hierarchical definitions and classification was criticized by De Villiers (1994) if it didn't enable promoting learners' conceptual understanding.

Considering the value of growing mathematics teachers professionally to secure the students' success (Morris et al., 2009), some recommendations to help those teachers handle their future classrooms could be presented. Practically, teacher education of geometry should sharpen the principle of economics/ minimality of a definition to allow prospective teachers to attain Van Hiele's level 3 since this minimality considers a universal principle that helps students developing more sophisticated reasoning when working deductively in mathematics (Avcu, 2022). The dynamic geometry software (e.g., Sketchpad, GeoGebra, Cabri) would help to advance learners' levels of geometric thinking while the discussion should be oriented towards (a) making the definition more economical through distinguishing between critical and non-critical aspects; and, at the same time, (b) explore each definition in terms of being excluded from or included in other related concepts.

Also, mathematics educators should train prospective teachers on how to track each student's progress evaluation; and adapt the pedagogical approaches while teaching quadrilaterals depending on the students' levels of study, wherein the uneconomical definitions might appear easier and more intuitive to young students than adults (Susanto, 2020; Van Dormolen & Zaslavsky, 2003). In that sense, a profound an argument on mathematical definitions embedded in school textbooks at different grades could be an essential issue during teacher education programs.

Suggestions for Future Research

Some limitations might be considered while interpreting the results of this study, specifically the developed framework. First and importantly, the characteristics of the study sample used to grow this framework; it was selected intentionally on the criteria of attaining Van Hiele's level 2 of geometric thinking, besides their knowledge of the school curriculum (including the topic of quadrilaterals). The second limitation is the small number of participants and the specified questions they responded to;

consequently, further refinements could be made to examine the framework validity with a large sample and variety of tasks. That is, when interpreting the results, we have to think deeply about Žilková's (2015, p. 31) notation of which, "the comparison of students' knowledge from different countries is extremely sensitive and it needs to be very carefully interpreted".

The present study, theoretically, is a perspective for further investigations to explore all sub-levels of Van Hiele's fixed levels; on the other hand, practically, it benefits reflections on proper pedagogical approaches and corresponding interventions in order to train prospective mathematics teachers to teach geometric thinking effectively.

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Declarations of Conflicting Interests

The authors declare no competing interests.

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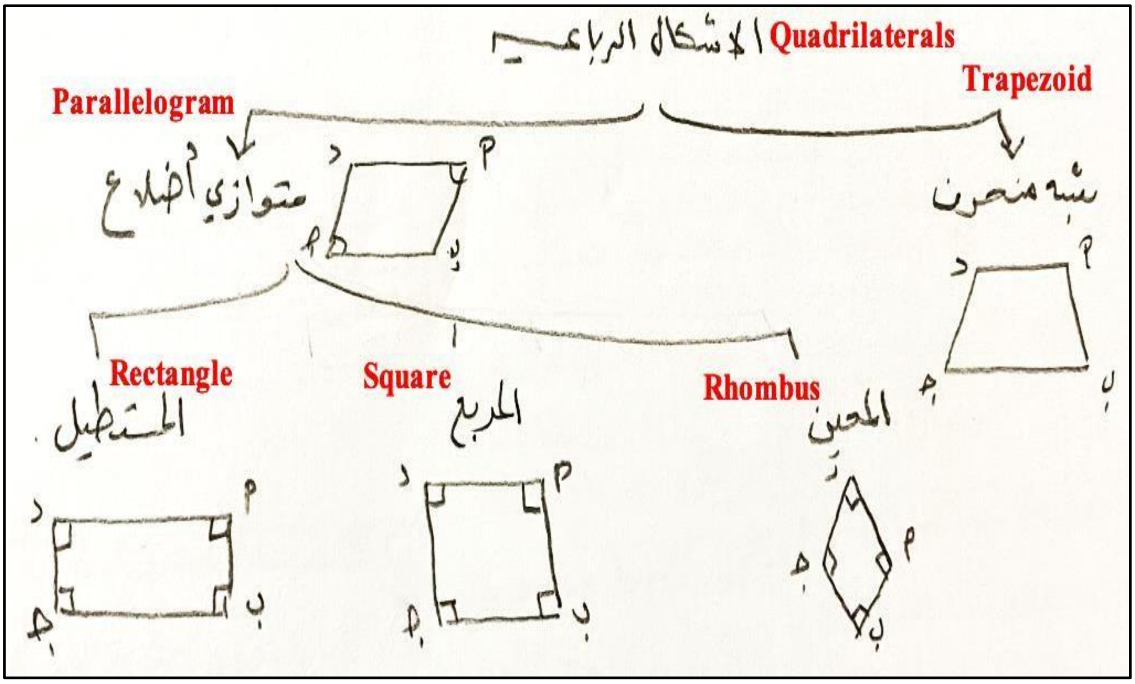
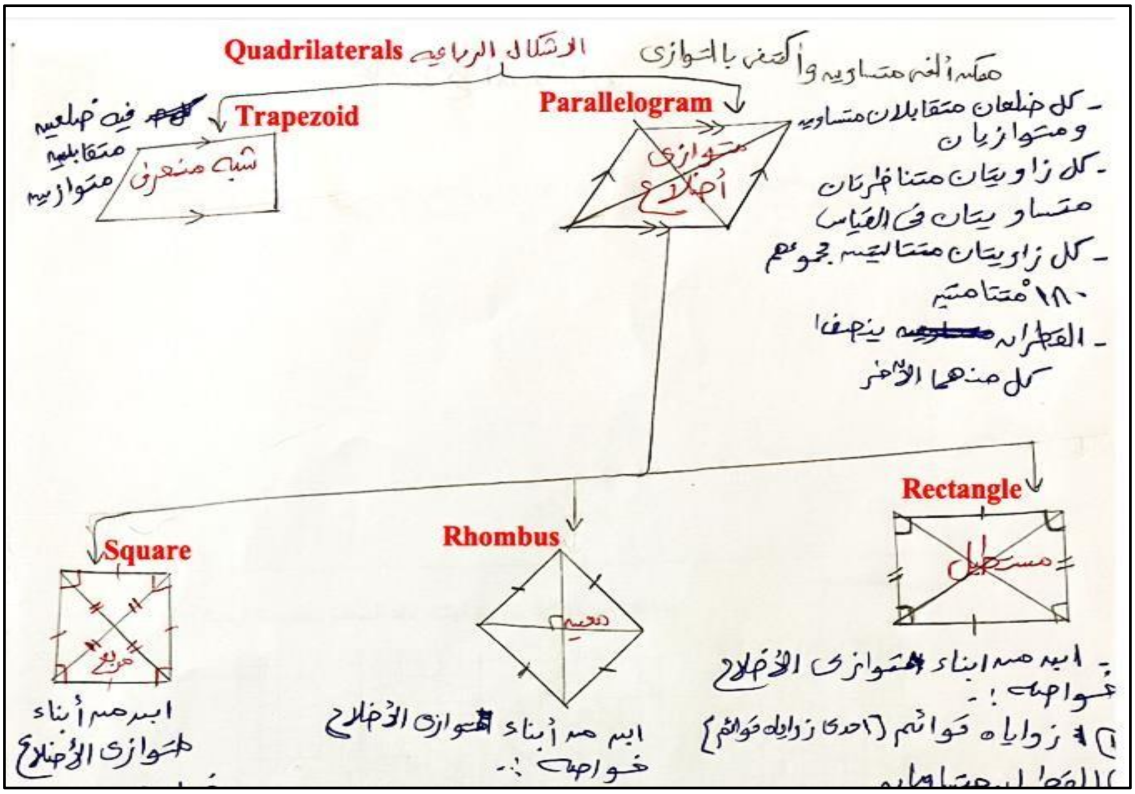
Appendix

An Example of Model A Representation of Relationships Among Special Quadrilaterals



An Example of Model B Representation of Relationships Among Special Quadrilaterals



Examples of Model C Representation of Relationships Among Special Quadrilaterals