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The Effectiveness of GeoGebra for Developing Mathematical Knowledge in Transformations of Functions and Graphs

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Abstract: This study aimed at assessing the effectiveness of GeoGebra while exploring how dynamic visualisation supports connections between symbolic rules and graphical behaviour, and examining students' engagement with GeoGebra applets during active learning within the SciMathUS bridging programme at Stellenbosch University in South Africa. A sequential explanatory mixed-methods design, a two-phase research approach where quantitative data were first collected through a pre-experimental design, after which the results were used to inform and conduct a second phase of qualitative data collection through a case study design guided by Yin's (2009) conditions for case selection to explain, interpret, or elaborate on the initial numerical findings. The sample for the study consisted of 48 students drawn from a cohort of 99, constituting a convenience sample. The intervention included active-learning tactics and GeoGebra-based activities across several computer-lab sessions. The data collected were analysed using paired t-tests and content-marking analysis. The quantitative findings from the analysis revealed statistically significant improvements ($p < 0.05$) with large-to-huge effect sizes (Cohen's $d = 0.95$–2.33), thereby confirming the rejection of the null hypotheses. The qualitative findings indicated that GeoGebra strengthened structural understanding by linking algebraic and graphical representations, enabling students to generalise transformation properties across various functions, including trigonometric ones. Students demonstrated more concise solution steps and greater confidence in visualising and articulating transformations. Demonstrated statistically significant improvements ($p < 0.05$) with substantial effect sizes, corroborating the rejection of null hypotheses. The study emphasises the importance of integrating dynamic technology into mathematics instruction to foster a deeper conceptual understanding and promote active learning. It recommends the broader adoption of GeoGebra as an instructional tool to address persistent challenges in learning function transformations. These findings affirm the educational benefits of dynamic mathematics environments and advocate for more participatory, technology-enhanced STEM education.

Keywords: GeoGebra, Mathematics, Knowledge, Transformations in functions and graphs, STEM.

Introduction

Mathematics is an important secondary school topic that offers students a unique opportunity to develop their analytical reasoning skills. At this level, mathematics becomes more structured and advanced, encompassing not only fundamental arithmetic but also algebra, geometry, statistics, and calculus. Current events demonstrate that mathematics is a vast field that encompasses reasoning, deduction, evidence, facts, measurements, and scientific observations. It also includes mathematical models of social systems, human behaviour, and natural phenomena. Therefore, mathematics is much more than just arithmetic and geometry. It is frequently seen as a tool for understanding the world around us and as a basic discipline in many other fields of study (Jayanthi, 2019; Yeasmin, 2017).

Mathematics is related to professional routes focused on science, technology, engineering, and mathematics (STEM). However, effective mathematics instruction is a problem affecting nations worldwide, not only South Africa (Glanvill-Miller, 2017; Moleko & Mosimege, 2020; Sun et al., 2025). Student achievement in mathematics and reading is also a concern for national and international organisations. Continuous endeavours are undertaken to mitigate these issues, as mathematics has emerged as the benchmark for assessing the educational and developmental health of nations. Moreso, academic achievement often includes numeracy or mathematics as assessment benchmarks, almost nationally and internationally (Fitzmaurice et al., 2021). These efforts aim to educate governments, schools, teachers, parents, and even students about the necessary changes in pedagogical approaches to enhance students' performance in mathematics. It is against this background that this study examined the effectiveness of GeoGebra for developing mathematical knowledge in transformations of functions and graphs.

Research Problem

Trends in record-keeping reveal that South African students struggle with curriculum concepts requiring deeper comprehension, particularly those involving sketching, drawing, and visualisation in the Mathematics National Senior Certificate examinations (DBE, 2013, 2014, 2015). Despite these challenges, most schools continue to adopt traditional teaching approaches that treat learners as passive consumers of knowledge (Westaway & Gravene, 2019). Research further indicates that many students cannot successfully solve mathematical problems (Lambrianos, 2019; Mabena et al., 2021). According to Wessels (2009), mathematics is often taught in isolation from real-life contexts, reducing it to mere computation and manipulation. Consequently, South African learners experience greater difficulty applying mathematics in real-world situations compared to their peers in other countries (Barnes & Venter, 2008). This study, therefore, examines the effectiveness of GeoGebra in teaching transformations of functions and graphs.

Students often struggle to visualise and generalise transformations of functions, such as vertical and horizontal shifts, reflections, and stretches. These concepts require learners to mentally manipulate graphs and predict changes in input-output relationships, which is challenging when relying solely on static representations. Without opportunities for dynamic interaction, students may fail to connect symbolic rules with graphical behaviour, leading to fragmented understanding. Dynamic software can bridge these conceptual gaps by providing real-time visual feedback and interactive tools that allow learners to experiment with transformations, observe patterns, and generalise principles across multiple examples. This active engagement supports deeper conceptual understanding and helps students internalise the underlying structure of functions rather than memorising isolated rules.

Traditional approaches to teaching function transformations typically rely on static representations, such as paper-based drawings or transparencies, which restrict students' ability to explore and understand the dynamic nature of mathematical objects. Faulkenberry and Faulkenberry (2010) illustrate that while static visualisations can demonstrate transformations like vertical or horizontal shifts, they do not allow learners to interact with graphs or observe real-time changes in input-output relationships. This lack of interactivity often results in students memorising transformation rules as isolated algorithms—a form of social knowledge — without developing a deep conceptual understanding (Kamii & Ewing, 1996; Lutz & Huitt, 2004). For example, students may learn that adding 3 to a function shifts its graph upward but fail to grasp the implications for critical points such as turning points and asymptotes.

Dynamic software, such as GeoGebra, addresses these limitations by enabling real-time manipulation of graphs while preserving their mathematical properties. Unlike static drawings, dynamic environments allow learners to experiment with multiple transformations, visualise changes in input-output values, and construct their own functions interactively. Sinclair and Yurita (2008) emphasise that dynamic representations preserve the characteristics of objects during transformation, offering opportunities for exploration that static methods cannot. This interactivity fosters active engagement, provides immediate feedback, and facilitates hypothesis testing, all of which are essential for meaningful learning. By shifting the focus from rote memorisation to conceptual understanding, dynamic software supports a constructivist approach to teaching transformations, aligning with the recommendations of Faulkenberry and Faulkenberry (2010) to emphasise input-output relationships rather than procedural rules.

Therefore, the research problem lies not in the use of paper-based activities themselves, but in their limitations compared to dynamic tools for fostering deeper conceptual understanding. While paper activities can support reasoning, dynamic software provides interactive, real-time visualisation that enhances students' ability to explore and internalise function transformations. Investigating the impact of dynamic software on students' comprehension of these transformations is critical to addressing this gap in mathematics education.

Research Focus

This study examined the effectiveness of GeoGebra in teaching mathematical transformations of functions and graphs.

Research Questions

1. Was there a substantial disparity in students' comprehension of function transformations and graphs using GeoGebra?
2. What are the learning gains for students following instruction on the transformation of functions and graphs utilising GeoGebra?

The first was translated into a two-tailed null hypothesis, $H_0 \mu$pre−test = μpost−test; $H_1 \mu$pre−test ≠ μpost−test as follows:

H_0 (null hypothesis): There is no significant difference in students' performance in function transformations using GeoGebra.

H_1 (alternate hypothesis) There is a significant difference in students' performance in function transformations using GeoGebra.

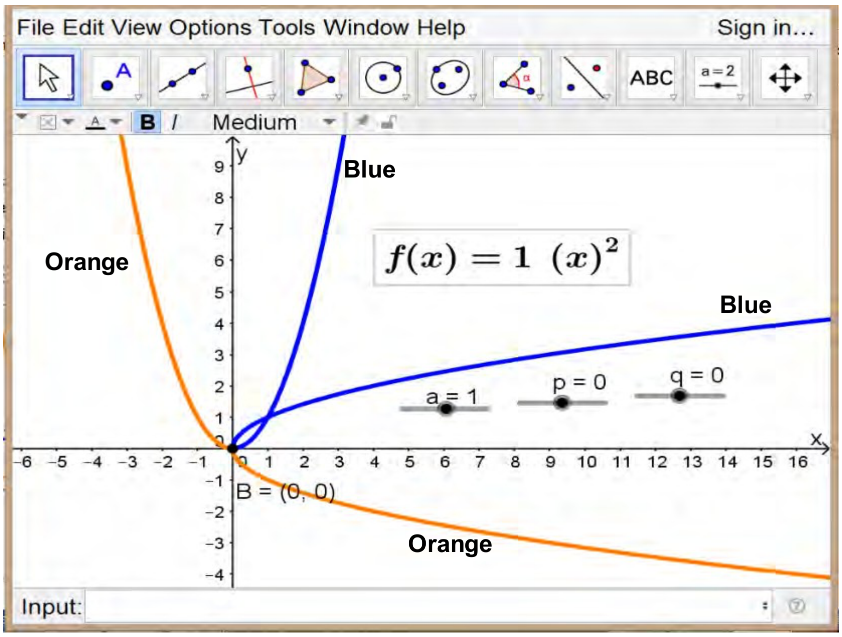
The second research question was non-testable and was answered qualitatively.

Literature Review

Digital devices, such as GeoGebra (a dynamic software), provide teachers with a far broader array of instructional options than chalkboard methods (Ndlovu *et al.*, 2011). The hands-free mathematics drawing differs from dynamic drawing in that it can be transformed while preserving the object's attributes (Sinclair & Yurita, 2008). This allows students to examine the figure or object from different angles. GeoGebra is an example of dynamic software that is explicit with its illustrations. The CAPS (DBE, 2011a) explains that students must draw the inverse graph while solving the inverse of a parabola, as shown in Figure 1.

Figure 1

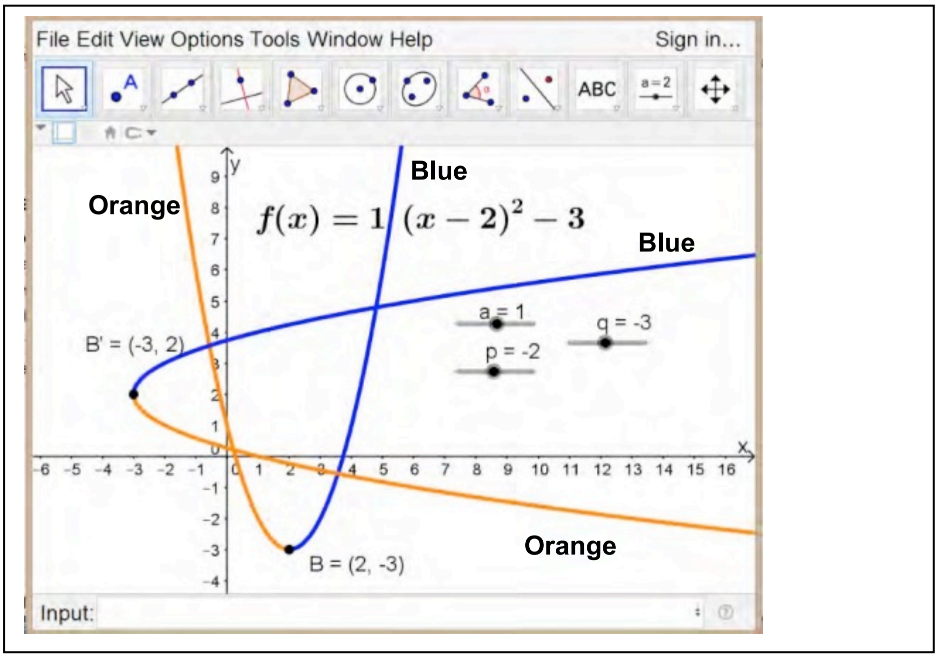
GeoGebra applet to teach inverse graphs.



Students are instructed to limit the scope of $f(x) = ax^2$ to guarantee the inverse is a function. The curriculum calls for changing the parameter a; however, p and q in $f(x) = a(x + p)^2 + q$ the inverse graphs must be zero, resulting in a domain of $x \leq 0$ or $x \geq 0$. Students might be able to memorise this information for the test, but they might not understand how to restrict the domain of a function to make the inverse a function, as illustrated in Figure 2.

Figure 2

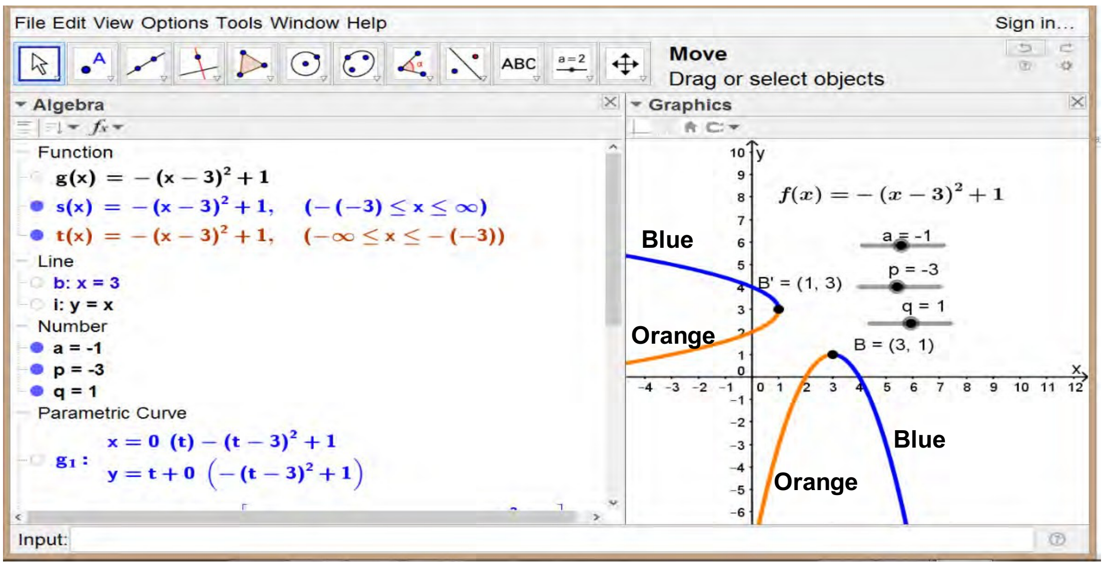
Restriction of $f(x)$ domain GeoGebra applet



By adjusting a, p, and q, students can see $f(x)$ in different ways. This helps them learn that the function's domain is determined by the x-value at the turning point and by related pedagogy. GeoGebra can also be thought of as a program that makes a dynamic geometry environment (DGE) (Pfeiffer, 2017; Qiang et al., 2025). Measurements in an algebra window change all the time, but with DGEs, the user can move parts of a geometry object and change the values of a, p, and q, as illustrated in Figure 3.

Figure 3

GeoGebra applet demonstrating the correlation between the graphical and algebraic representations



GeoGebra's algebraic and visual windows help one see variations in the graph's forms and equations. Figure 3's applet displays another orientation than Figure 2's. DGEs provide an interactive space for algebra and geometry visualising and exploration. GeoGebra boasts DGE traits. Serving not only geometry but also algebra, statistics, and other disciplines.

It is essential to be able to graph functions; the DBE (2011b; 2014) reported that understanding functions is one of the key areas where students struggle. Plotting points to make graphs can be time-consuming. However, using math sketching tools like GeoGebra and GeomBot can make this process much faster (Baccaglini-Frank et al., 2020). If technology facilitates the creation of graphs in a tidy, rapid, effortless, and precise manner, students can concentrate on the more complex task of analysing functions and graphs, allowing them additional time for independent exploration. While pen-and-paper tasks can examine the impact of parameters across various functions, GeoGebra enables students to engage in a significantly greater number of activities in the same time. Dynamic mathematics software facilitates the manipulation of graphs through stretching, shifting, and reflecting, being an important functionality and requirement for visualisation (DBE, 2011a).

Effective teaching, particularly in mathematics, requires the readiness to use diverse teaching resources and strategies, necessitating a blended learning approach that integrates various methods and resources in a technology-driven era (Luzano, 2025; Oladele, 2024; Tabach & Trgalová, 2020). This study used the right information and skills to investigate the feasibility of using GeoGebra in maths classes with the participants. This study used technology together with both active learning and traditional teaching methods. GeoGebra was used to teach and learn in both standard and hands-on ways, as were dynamic math programs like Cabri and Geometer's Sketchpad to aid learning. The study focused on the following transformations of functions: reflections in the x-axis, vertical stretching, and translation.

Theoretical Frameworks

This study employed Piaget's theory of knowledge, the Instrumentation component of the Theory of Instrumental Genesis (TIG), and the Process of Coherence of Formation as its theoretical framework to investigate how pupils progress to higher levels of abstraction when they change functions and graphs.

Piaget's hypothesis of several types of knowledge: Piaget (1952) posits that three kinds of knowledge exist, namely, social (conventional), physical (corporeal), and logico-mathematical. He also discusses how pupils learn each type. People learn social skills over time (Kamii, 2014), and in the era of education 4.0, they can only acquire these skills by interacting with others (Susanto, 2024; Zhou & Schofield, 2024). DGE and other hands-on activities help students learn about the world around them. Logico-mathematical knowledge is a form of abstract thinking applicable in contexts outside direct interaction with a tangible stimulus. Using GeoGebra to explore concepts can help students understand the physical world, a foundational principle for logical thinking essential in mathematics learning (Piaget, 1952). It lets you make generalisations and abstractions (Lutz & Huitt, 2004). Students made guesses in GeoGebra by using generalisations, which then led to the creation of generalisations and abstractions. Because of this, the study also examined how manipulating GeoGebra might help students develop both physical and logical-mathematical understanding of changing functions and graphs.

Instrumental genesis (IG) is a learning theory that stresses how artefacts evolve and transform into instruments through user usage and knowledge. Moraga et al. (2025) explained that instrumental genesis pertains to the use of artefacts that facilitate knowledge construction processes in mathematical lessons. Instrumental Genesis explains that an artefact becomes an instrument through two interconnected processes: instrumentation and instrumentalisation. Instrumentation refers to how the artefact influences the user’s actions and cognitive structures, shaping their approach to tasks and understanding of concepts. In contrast, instrumentalisation involves the user adapting and customising the artefact to serve their own objectives (Vérillon & Rabardel, 1995). Instrumentation refers to the process by which the user shapes the item using their knowledge and past experiences. The artefact's limits and potentialities shape the user (Fahlgren, 2015). Instrumentation is defined as the act of mastering tools by the artefact user while enabling the construction of activities under certain restrictions (Ndlovu et al., 2011).

Furthermore, instrumentalisation involves discovery and selection of relevant functionalities, customisation, and artefact change, sometimes in unforeseen directions by the designer. Using GeoGebra, the study demonstrated that students' learning gains can be enhanced by learning technology. Recent research continues to apply and extend this theoretical lens to digital learning environments, highlighting how technological tools afford opportunities for instrumental genesis in mathematics classrooms (Dennis & Buchbinder, 2022)

The third conceptual lens guiding this research is the process of coherence formation, which explains how learners integrate multiple representations into a unified understanding. This process is closely aligned with Gentner's (1983) structure-mapping theory, originally developed to account for analogical reasoning but later adapted to describe representational linking (Seufert & Brünken, 2004). From a cognitive perspective, Seufert and Brünken (2004) stress the importance of understanding how students coordinate and reconcile diverse representations during learning. Because many mathematical concepts, such as transformations of functions, are highly abstract, teachers often employ analogies and dynamic tools to support conceptual comprehension (Amir-Mofidi, Amiripour, & Bijan-Zadeh, 2012). In this study, GeoGebra was used to visualise different representations — graphical, algebraic, and tabular — and to demonstrate the notation and effects of function transformations explicitly. Coherence formation involves a sequence of cognitive operations: identifying relevant elements, mapping structural correspondences, and synthesising these into an integrated mental model. This alignment of relational structures promotes effective reasoning, transfer, and deeper understanding of mathematical relationships (Gentner, 1983; Seufert & Brünken, 2004).

Materials and Methods

Design

This study adopted a pragmatic paradigm and employed a sequential exploratory mixed-methods design to investigate the effectiveness of GeoGebra-based instruction among 48 students enrolled in the year-long SciMathUS bridging programme at Stellenbosch University. The mixed-methods approach was selected for its ability to integrate quantitative and qualitative data, providing comprehensive insights that prioritise practical solutions over philosophical alignment (Creswell, 2003; Taheri & Okumus, 2024). This study is contextualised within the Stellenbosch University SciMathUS Preparation Programme, which helps students from disadvantaged backgrounds improve their Mathematics and Science marks while also gaining academic and computer literacy, thereby qualifying for higher education. It offers a second chance to those who narrowly missed meeting the entry requirements. The Centre offers it for Pedagogy (SUNCEP) and focuses on practical skills, hard work, and the utilisation of resources to build future STEM leaders. The intervention spanned seven months, during which various mathematical topics were taught using GeoGebra. Within this timeframe, the teaching of function transformations was allocated a one-week segment of the curriculum. The intervention was implemented in the computer lab over three consecutive days, with six 50-minute sessions per day. GeoGebra was integrated specifically to support the learning of function transformations. Students interacted with prepared GeoGebra applets and also typed in the equations of the functions given to them from a worksheet in a computer lab environment designed to facilitate conceptual exploration. In the Graphic View, they observed how graphs shift, reflect, stretch, or compress, while in the Algebraic View, they examined the corresponding symbolic notation of the transformed functions.

Data collection for the topic of function transformations followed a structured sequence. The quantitative component began with a pre-test to assess prior knowledge. During the intervention, observations were conducted while students engaged with GeoGebra-based activities. Following the instructional sessions, a post-test was administered to measure learning gains. Both the pre- and post-tests were analysed quantitatively to determine performance changes and qualitatively to examine reasoning patterns and conceptual development, providing insight into knowledge gains.

The qualitative component employed naturalistic observation to examine events in real-world settings without manipulating the learning environment, thereby substantiating conceptualisations derived from participant-provided data (Harwell, 2011). Students' problem-solving skills in function transformations were closely monitored during computer lab sessions before and after the intervention to assess learning progress. The quantitative component employed a quasi-experimental design due to the absence of strict control over external factors. Due to ethical considerations, the design did not include a control group, as all students received similar instruction within the regular SciMathUS programme. To mitigate maturation effects — which pose a threat to internal validity when extended intervals occur between pre- and post-tests — the intervention was condensed into three consecutive days (Behar-Horenstein & Niu, 2011).

The integration of quantitative and qualitative methods yielded a sequential explanatory mixed-methods design, aligning with scholarly recommendations. Taheri and Okumus (2024) emphasise that mixed-methods research and the integration of diverse data sources are critical for advancing the social sciences, particularly in educational research. Similarly, Creswell (2003) notes that mixed-methods approaches employ concurrent or sequential data collection strategies to comprehensively address research questions. These perspectives informed the decision to adopt this approach, which was deemed suitable for the study.

Sample and Participants

The study participants were a convenience sample of 48 students from two of the four classes, for a total population of 99 students enrolled at SciMathUS. Although the 48 students were enrolled in two separate classes, it was not feasible to divide them into control and experimental groups because they attended a combined class every third day. This scheduling arrangement made it impossible to implement different instructional approaches for the two groups in the SciMathUS bridging programme at a South African university. Through this year-long program, educationally disadvantaged students, socially, geographically, and economically, are allowed to remediate their Grade 12 marks to gain admission into University STEM (Science, Technology, Engineering, and Math) programs. SciMathUS's all-encompassing curriculum includes real-world skills that support learning and critical thinking in problem-solving (Malan et al., 2014; Oladele et al., 2022).

Instrument and Procedure

The instruments used in the study were the pre- and post-tests, as well as a researcher-designed observational scoring rubric. The Pre and post-tests was a standard set of questions presented to the study participants with the aim of ascertaining (achievement, ability, or aptitude) and administered using the paper-and-pencil mode based on which the null hypothesis that the experimental groups' post-test scores would not be different from their pre-test scores when it comes to showing how much the participants knew about transformation of functions in math was tested.

The pre- and post-tests have the following domain-based questions:

Domain-1: Deriving the equation of a transformed graph from the given algebraic notations

Domain -2: Calculating the critical values (turning points, asymptotes, etc.) of the transformed function from the given algebraic notations

Domain-3: Explanation of the transformation in words from the given algebraic notations; and

Domain-4: Sketching the transformed graph of the cosine graph from the given algebraic notations

To ensure the quality of the study instruments, the researcher tested them with 10th-grade math students, who were representative of the study population. The validity of the instrument was determined by face and content validity (Cook & Beckman, 2006; Kimberlin & Winterstein, 2008). Two experts who have been involved in secondary school mathematics teaching for over two decades validated both the pre- and post-test content. Additionally, an expert with over five years of experience in Grade 12 mathematics assessment was engaged to validate the scoring rubric used for the pre- and post-tests. To ensure the reliability of the instruments, the inter-rater reliability of the scores for both tests was analysed using intra-class correlation (ICC) (Eagan et al., 2020; Field, 2013). A reliability coefficient of 0.987 was obtained, which was considered acceptable, given the benchmark of 0.75 at the 0.05 significance level (Morgan et al., 2013; Waninge et al., 2011). This result indicates no substantial variations between engaged expert scores.

The observatory scoring rubric contained descriptors to assign marks, ensuring consistency in evaluating students' responses. If a student erred in a question and subsequently utilised the erroneous information in the following question or proceeded to build upon the mistake, it was assessed favourably. The DBE designates this as consistent correctness in the notes, and it is employed in the assessment of the NSC test.

Procedure for Data Collection

The teaching of function transformation was conducted in the computer labs, which favoured a carefully guided, coordinated approach in which the teacher moved around the classroom. At the same time, students used computers with the GeoGebra application installed, and the teacher supervised and answered questions as needed (Drijvers et al., 2013). Also, technical demo, explain-the-screen, link-the-screen, and discuss-the-screen orchestration were also observed, while students worked autonomously. Before the intervention program, participants completed pre-tests, and after it, they took post-tests to see how their function transformation learning had improved in terms of:

1. comprehending how algebraic notation alterations, such as $f(x-2)-3$, affect functions or graphs

2. utilising a modified function's nomenclature to determine its equation

3. sketching the function's graph when presented with the transformation in words or notation; and

4. identifying the most important parts of the graphs or functions.

Ethical Considerations: The Western Cape Education Department granted written authorisation for research to be conducted in two schools. The research conducted in two schools served as the study's piloting phase and focused exclusively on function transformations. The piloting at the schools focused solely on function transformations, although it provided valuable insights for the primary research, as SciMathUS students are simultaneously undertaking the NSC tests. The pilot data informed the necessary modifications for conducting the study with the SciMathUS students at the university. Ethical approval was obtained from the relevant university authorities to conduct this study with students enrolled in the SciMathUS bridging programme. The permission and ethical certificate can be obtained upon request. In deploying the study, active learning and interactive approaches were employed. The researcher's distinctive pedagogical method involved exploring mathematics through GeoGebra and curated activities. The study participants were explicitly informed of the teaching methods used to ensure informed consent while maintaining anonymity and confidentiality.

Data Analysis

This study's qualitative data analysis focused on students' learning improvements following instruction in function and graph transformations using GeoGebra. The qualitative analysis of student replies in the pre- and post-test facilitated the determination of their learning gains using students' t parametric statistics to evaluate the null hypothesis (H_0). A statistically significant value was determined with a p-value of 0.05.

Results

Research Question 1: Was there a substantial disparity in students' comprehension of function transformations and graphs using GeoGebra?

To investigate whether students' post-test answers differed significantly, the research question was translated into a testable hypothesis stating that there is no significant difference in students' performance in function transformations using GeoGebra, as a quantitative reflection of their learning gains in transformations of functions. A pre-test assessed students' prior knowledge from their school years before the intervention, while the post-test was administered after the intervention sessions in the computer lab. The results are shown in Table 1.

Table 1

Paired sample t-test shows the students' means of the pre- and post-test in the different domains

	Paired Differences					t	df	Sig. (2-tailed)	Cohen's d	Effect size
	Mean difference	SD	SE	95% Confidence Interval of the Difference						
				Lower	Upper					
Pre- & post-domain1	33.333	20.456	2.953	27.394	39.273	11.290	47	.000	1.69	Very large
Pre- & post-domain2	36.625	17.663	2.549	31.496	41.754	14.366	47	.000	2.05	Very large
Pre- & post-domain3	42.521	25.017	3.611	35.257	49.785	11.776	47	.000	1.95	Very large
Pre- & post-domain4	27.417	25.618	3.698	19.978	34.855	7.415	47	.000	0.95	Large

Table 1 shows that every p-value (0.000) is less than the α-value (0.05). Therefore, the null hypotheses (H_0) that stated there was no significant difference in students' performance using GeoGebra in learning transformations of functions were rejected in favour of the alternative hypothesis. It is important to note that in all the areas listed above, the post-test means were statistically significantly higher than those for the pre-test. Furthermore, the effect sizes are substantial across three areas. Despite the absence of a comparison group for this study, the results indicate that the students enhanced their comprehension of function transformations. The average differences across all domains indicate that students' comprehension of higher-order questions regarding function transformations has improved. The national diagnostic results from the DBE (2013; 2014) indicate that students lack comprehension of functions. Nonetheless, it was encouraging to observe students advancing their comprehension of drawing the transformed graphs of trigonometric functions without formal instruction on this topic. This study shows that it can enhance the comprehension of function transformations.

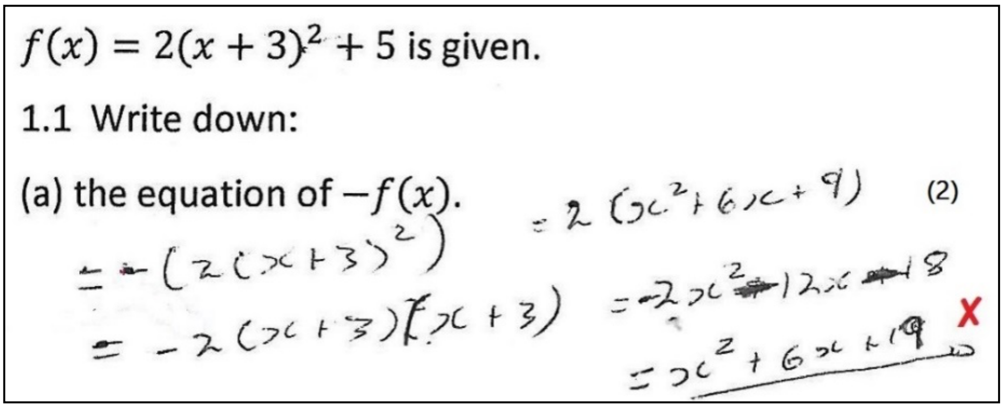
Research Question 2: What are the learning gains for students following instruction on the transformation of functions and graphs utilising GeoGebra?

Selected responses from students' pre- and post-tests illustrate the learning gains achieved following instruction with GeoGebra

Figure 4 illustrates Student 1's approach, where the first step involved simplifying the term within the parentheses. After this, the student applied the reflection transformation indicated by $-f(x)$. The original function given in the question was $f(x) = 2(x+3)^2 + 5$.

Figure 4

Response of Student 1 to the pre-test question on the equation of $-f(x)$


$f(x) = 2(x+3)^2 + 5$ is given.

1.1 Write down:

(a) the equation of $-f(x)$. (2)

$= -(2(x+3)^2)$

$= -2(x+3)(x+3)$

$= 2(x^2+6x+9)$

$= -2x^2 - 12x - 18$

$= x^2 + 6x + 19$

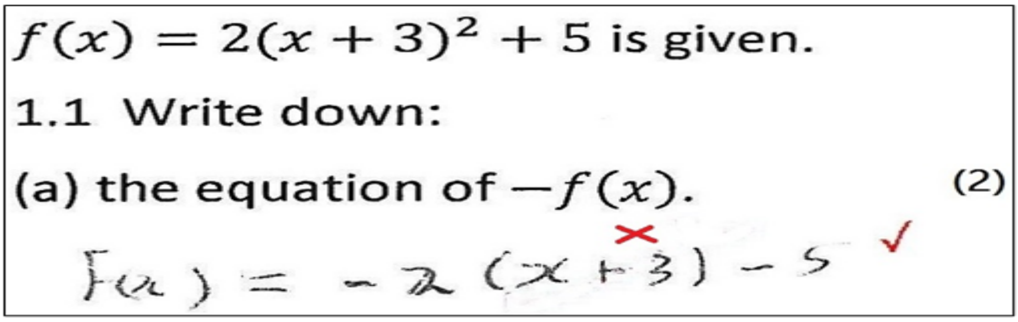
X

Student initially simplified the parentheses and thereafter executed the transformation of $-f(x)$. The function provided to the students is $f(x) = 2(x + 3)^2 + 5$. Students' answer to the pre-test question about the equation of $-f(x)$, which evaluated their algebraic manipulation skills, is worth two points. The student's initial mistake was omitting 5 from his expression, despite his understanding that the sign change pertained to the full statement, as demonstrated in step 4. The second error the student made was dividing the expression by -2, incorrectly treating the expression $-2x^2 - 12x - 18$as if it were an equation. . As a result, the student received no points for this question. The student used a more protracted technique to resolve this difficulty without success. The student failed to respond to question 1.1(c), which requested a verbal explanation of the transformation. The student missed the significance of $-f(x)$ but partially grasped the notation of $-f(x)$ by multiplying by -1, albeit failed to incorporate 5

The concept tested here was the algebraic manipulation of $-f(x)$ and two marks were allocated for this question. The student's first error was excluding 5 from his expression, but he knew that the change in sign applied to the entire expression, as shown in step 4. The student's second error was to divide by -2, because he equated the expression $-2x^2 - 12x - 18$ equal to zero. Consequently, the student scored no marks in this question. The student attempted a longer method to solve this problem unsuccessfully. The student did not answer question 1.1(c), which asked to explain the transformation in words. The student thus did not understand the meaning of $-f(x)$ was but understood the notation of partially $-f(x)$ by multiplying by -1, but did not include 5. Student TF1 entered the SciMathUS programme with 34% in NSC Mathematics. He scored 9% in the pre-test and 39% in the post-test for transformation in functions.

Figure 5

Response of Student 1 to the post-test question on the equation of $-f(x)$



$f(x) = 2(x+3)^2 + 5$ is given.

1.1 Write down:

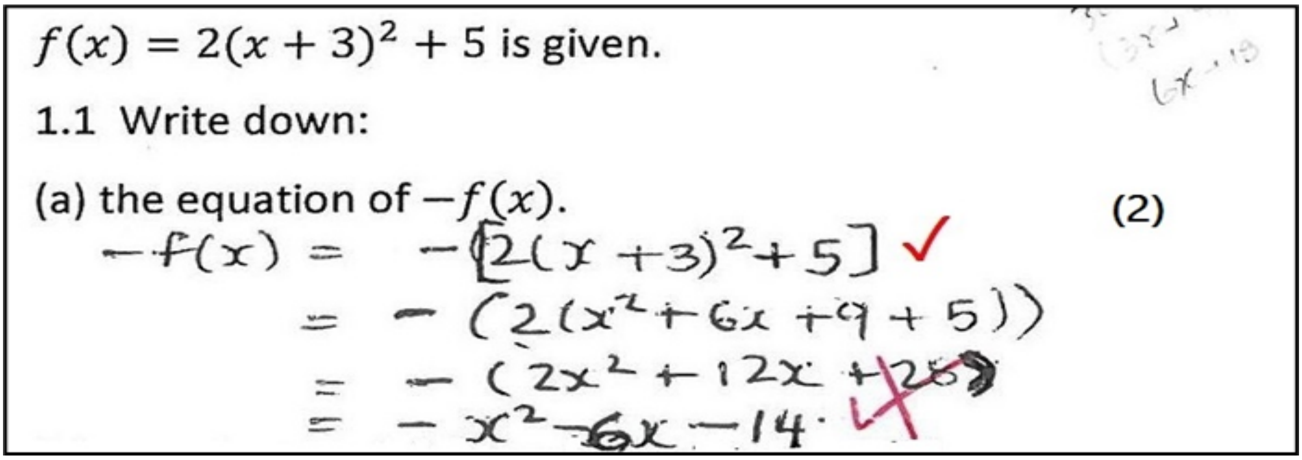
(a) the equation of $-f(x)$. (2)

$F(x) = -2(x+3) - 5$

As shown in Figure 5, the student achieved a mark of -5 but lost an additional mark due to an oversight in failing to raise the brackets to the second power. In the post-test, the student demonstrated a more streamlined approach, using fewer steps to reach the answer compared to the pre-test, as shown in Figure 4. Overall, the student scored 9% on the pre-test and 39% on the post-test for transformations of functions. The student demonstrated a comprehensive understanding of the notation used in the algebraic view of GeoGebra, reflecting their effective engagement with the instructional material. This engagement suggests that the student not only absorbed the information presented but also accurately interpreted the algebraic concepts and their applications within the software environment.

Figure 6

Response of Student 2 to the pre-test question on the equation - $f(x)$



$f(x) = 2(x+3)^2 + 5$ is given.

1.1 Write down:

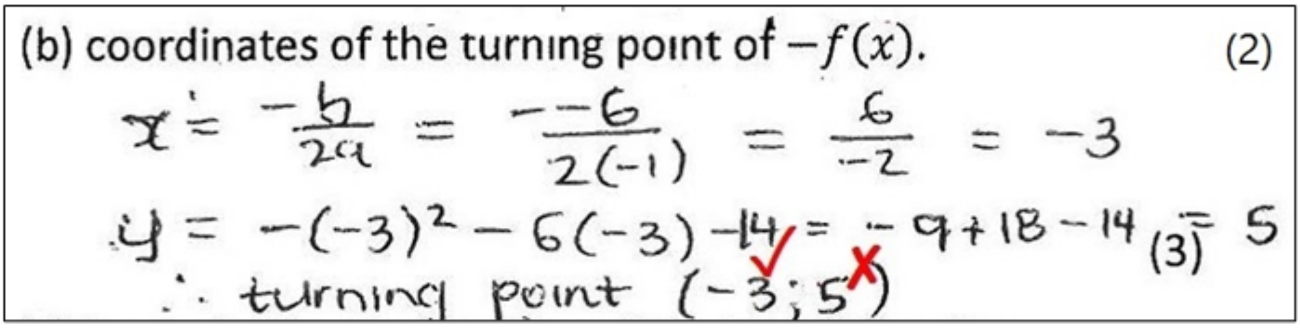
(a) the equation of $-f(x)$. (2)

$$-f(x) = -[2(x+3)^2+5]$$
$$= -(2(x^2+6x+9+5))$$
$$= -(2x^2+12x+28)$$
$$= -x^2-6x-14$$

The student began by multiplying f(x) by -1. The student commenced with the appropriate methodology but then simplified the brackets, resulting in an erroneous conclusion. The student additionally regarded the function as an equation and divided it by 2. The student commenced the program with a 69% score in the NSC, indicating that she achieved excellent results in NSC Mathematics; however, she was unable to recall the necessary procedures. This indicates that despite achieving good grades in NSC Mathematics, the student still equated the formula $-(2x^2 + 12x + 28$ to zero, treating the expression as an equation. Based on the student's response, it may be inferred that she perceived the functions -$f(x) = -(2x^2 + 12x + 28)$ and $-f(x) = -x^2 - 6x - 14$ as equivalent. Figure 6 illustrates the response of a student to a pre-test question, where the turning point is calculated as (-3, 5), provided with the hint: f(x) = 2(x + 3) ^2 + 5.

Figure 6

Response given by Student 2 in the pre-test regarding the turning point or vertex of - $f(x)$



(b) coordinates of the turning point of $-f(x)$. (2)

$x = -\frac{b}{2a} = -\frac{-6}{2(-1)} = \frac{6}{-2} = -3$

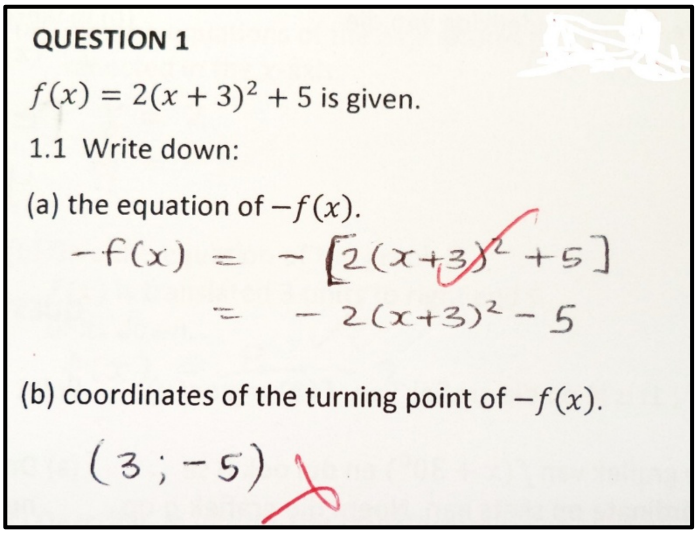
$y = -(-3)^2 - 6(-3) - 14 = -9 + 18 - 14 = 5$ (3)

$\therefore$ turning point $(-3; 5)$

As shown in Figure 6, a student committed a calculation error by misadding, but proceeded with the question as illustrated to provide the accurate reason for the transformation. The objective evaluated was whether the students could extract the coordinates of the turning point as derived from the equation $f(x) = a(x + p)^2 + q$. They might also determine the pivot-turning point from $f(x) = ax^2 + bx + c$ using the formula $x = -\frac{b}{2a}$. Efforts were geared towards determining whether students could ascertain from the function $2a\ f(x) = 2(x + 3)^2 + 5$ that the turning point (vertex) is located at (-3, 5), and whether, when enquiring about -$f(x)$, they recognised that the negation of $f(x)$ at (-3, 5) corresponds to (-3, -5). This result, as shown in Figure 7, indicates that Student 2 modified her methodology for responding to question 1(b) in the post-test.

Figure 7

Response given by Student 2 in the post-test, the equation of -f(x) and the turning point or vertex of - $f(x)$



QUESTION 1

$f(x) = 2(x + 3)^2 + 5$ is given.

1.1 Write down:

(a) the equation of $-f(x)$.

$-f(x) = -[2(x+3)^2 + 5]$

$= -2(x+3)^2 - 5$

(b) coordinates of the turning point of $-f(x)$.

$(3; -5)$

The student recited the vertex of (x) as (3, -5). Despite the student's erroneous answer, the solution was derived without any calculations as shown in Figure 8.

Figure 8

Pre-test responses of a student regarding transformation type

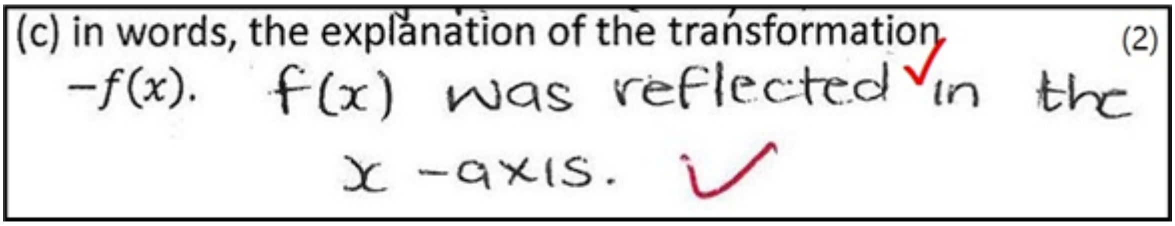
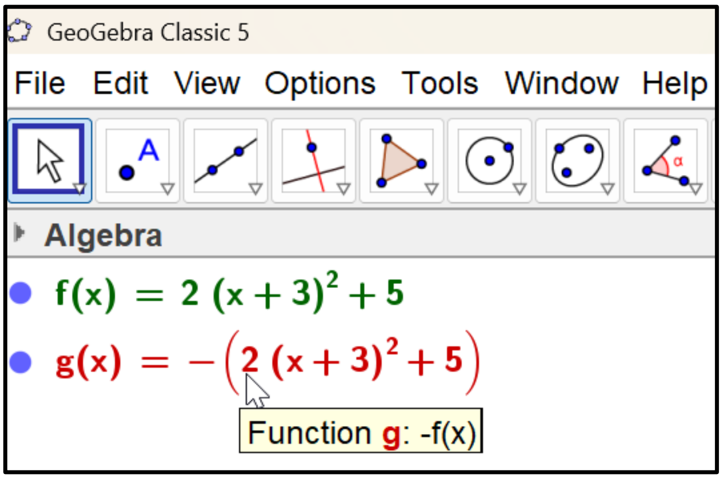

(c) in words, the explanation of the transformation (2)
$-f(x)$. f(x) was reflected in the
x-axis.

Figure 8 illustrates a student's responses to the transformation type of $-f(x)$ in the pre-test for Student 2. She responded the same in the post-test, indicating that the student understood that $-f(x)$ signifies reflection across the x-axis and could calculate the turning point and elucidate the transformation; nevertheless, she failed to recognise that the vertex of $-f(x)$ should be (-3; -5). This indicates that the student did not comprehensively grasp the features of reflection in a quadratic function and was likely instructed merely to recall the reflection from the notation $-f(x)$. GeoGebra applets were used to demonstrate the impact of transformations on input and output values to students. In the post-test, a student replicated the procedures from the pre-test but omitted the first simplification of the brackets in the post-test, as illustrated in Figure 7. The way that the students answered this question in the post-test shows how GeoGebra helps for coherence formation on deeper structure level of the notation $-f(x)$ (Seufert & Brünken, 2004).

As shown in Figure 8, GeoGebra Algebraic window demonstrate the same method used by two of the study participants in solving the problem. Furthermore, students' responses to this question in the post-test demonstrate how GeoGebra facilitates the establishment of coherence at a deeper structural level of the notation $-f(x)$ as shown in Figure 9. In GeoGebra, hovering over or double-clicking on *g(x)* displays the command that was entered. During instruction on function transformations, students were actively engaged by typing the equations for *f(x)* and *–f(x)* themselves. The transformation *–f(x)* was represented in GeoGebra using the notation *g(x)*.

Figure 9

Modification display in the GeoGebra Algebra window

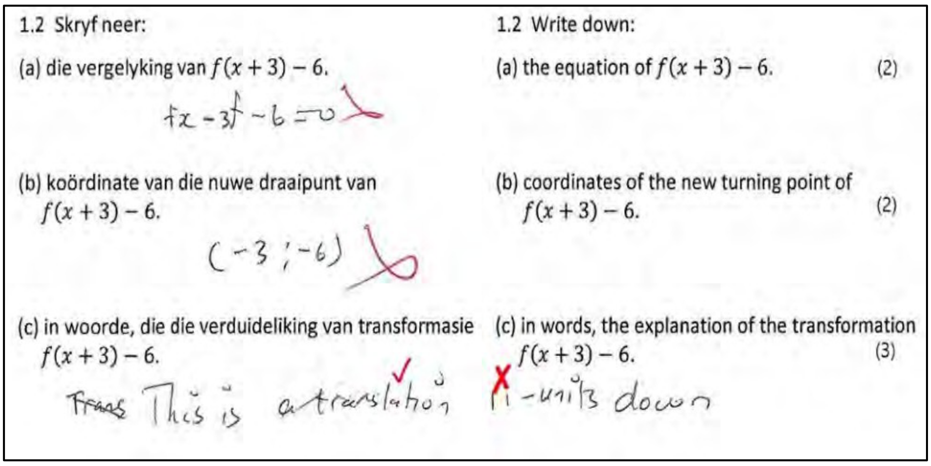


During the treatment phase, students used GeoGebra for five hours to learn about how functions change. Students examined polygon reflection, translation, rotation, and dilation before changing functions using GeoGebra tasks designed by the researcher. They used GeoGebra applets produced by the teacher-researcher and constructed their own activities to study function transformations.

Figure 10 illustrates a deficiency in understanding the derivation of the equation; yet Student 3 acknowledged that the transformation represented a translation in the pre-test.

Figure 10

The student's answer to the pre-test question on $f(x + 3) - 6$



1.2 Skryf neer:

(a) die vergelyking van $f(x + 3) - 6$.

fx - 3f - 6 = 0

(b) koördinate van die nuwe draaipunt van $f(x + 3) - 6$.

(-3 ; -6)

(c) in woorde, die die verduideliking van transformasie $f(x + 3) - 6$.

1.2 Write down:

(a) the equation of $f(x + 3) - 6$. (2)

(b) coordinates of the new turning point of $f(x + 3) - 6$. (2)

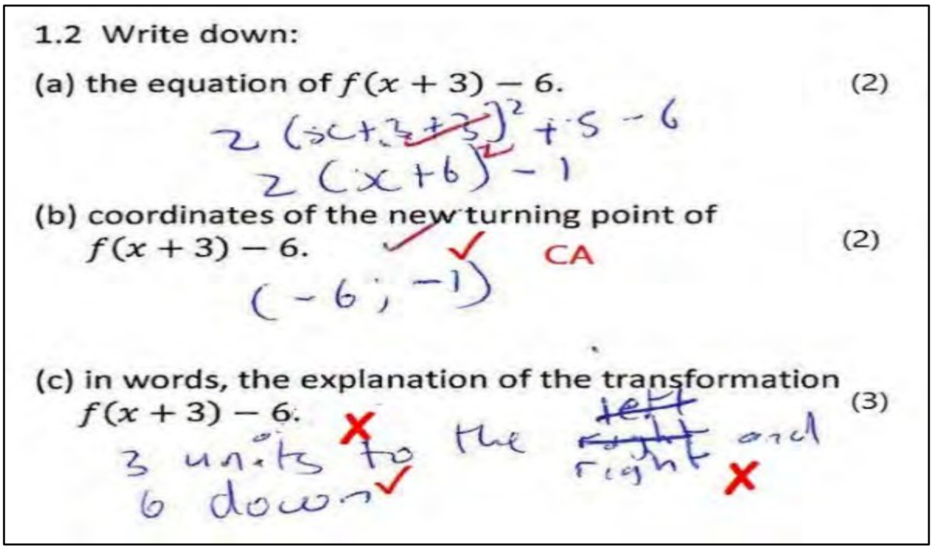
(c) in words, the explanation of the transformation $f(x + 3) - 6$. (3)

~~Trans~~ This is a translation ✓ ✗ 1-units down

A question was asked about whether students could complete the algebraic manipulation of $f(x + 3) - 6$. This question received two marks for parts (a) and (b) and three points for part (c). A student only got one mark because he realised that $f(x + 3) - 6$ signifies a translation. The student's answer to the post-test question is shown in Figure 11.

Figure 11

Student 3's post-test answers to question $f(x + 3) - 6$



1.2 Write down:

(a) the equation of $f(x + 3) - 6$. (2)

$2(x + \cancel{3} + 3)^2 + 5 - 6$

$2(x + 6)^2 - 1$

(b) coordinates of the new turning point of $f(x + 3) - 6$. (2)

✓ ✓ CA

(-6 ; -1)

(c) in words, the explanation of the transformation $f(x + 3) - 6$. (3)

3 units ✗ to the ~~left~~ ~~right~~ right ✗ and 6 down ✓

The student demonstrated comprehension of vertical translation but encountered difficulties with horizontal translation. The learner exhibited significant confusion over the horizontal translation. The horizontal translation was indicated as three units to the right and received one mark out of three for a downward movement of six units. This display indicates that the student perceived $f(x + 3)$ as a move to the right on the horizontal axis.

Students used GeoGebra to explore the relationship between input and output values, illustrating the meaning of the notation $f(x+3)$. The figure demonstrates that the original vertex at (4 ;–3) is translated three units to the left, resulting in a new vertex at (1; –3).

Figure 12

$f(x+3)$ notation explanation

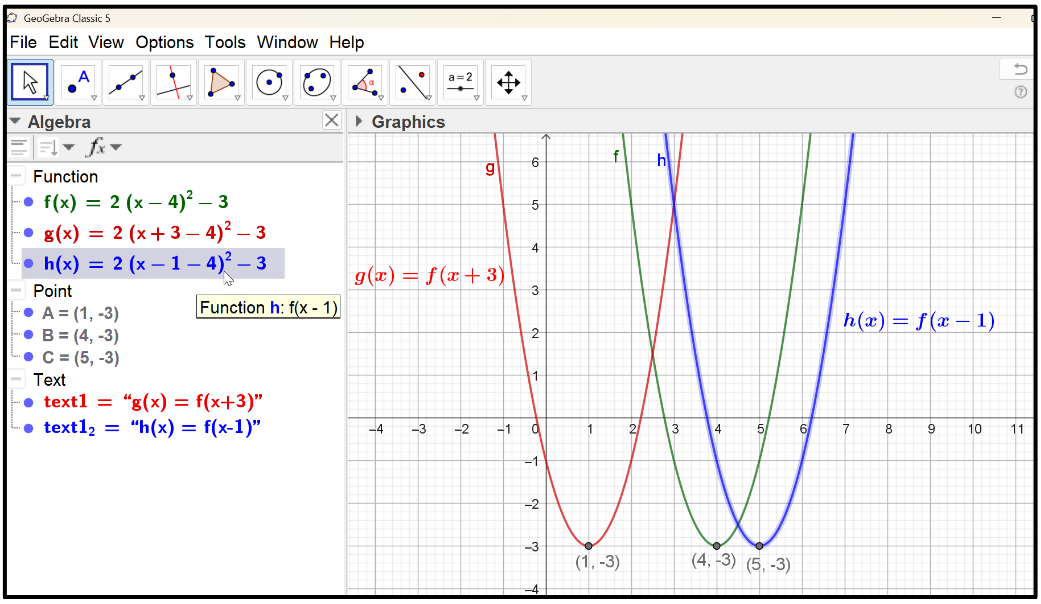
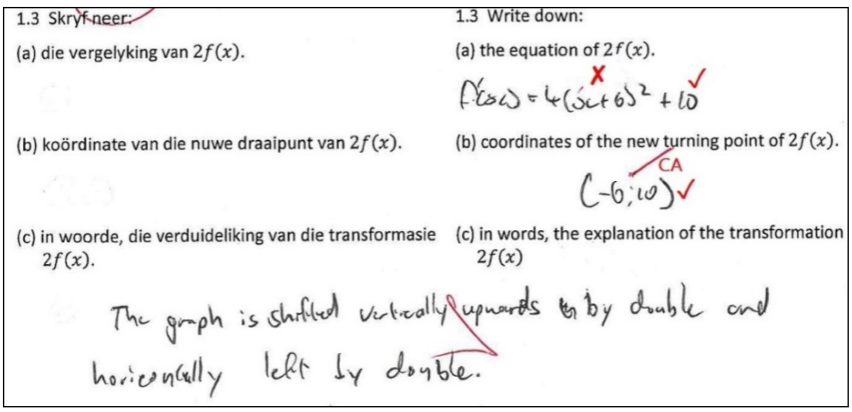


Figure 12 suggests that the student may have been attempting to recall the methods but exhibited confusion over the direction of the graph's movement, specifically whether it was moving left or right. During class discussions and interviews, some students indicated that, in school, they were primarily taught to memorize formulas and procedures rather than develop conceptual understanding. This approach often results in rote manipulation of symbols and memorization of algorithms (Faulkenberry & Faulkenberry, 2010), reflecting the transmission of social knowledge (Piaget, 1952; Kamii, 2014) rather than meaningful learning. One of the key challenges encountered in this study was facilitating the unlearning of these memorisation-based methods among SciMathUS students.

Figure 13 shows that Student 4 demonstrated in the pre-test, that the student understood the necessity to multiply $f(x)$ by 2, but provided an erroneous solution by defining f(x) = 2(x + 3)^2 + 5.

Figure 13

Student 4's pre-test answers to question 1.3



Student 4 reported that the turning point of $2f(x)$is $(-6\,;10)$ in the pre-test. This response was marked correct because the student accurately interpreted the incorrect answer provided in Question 1.3b. The student demonstrated procedural knowledge by computing the coordinates of the turning point using the format $f(x) = a(x+p)^2 + q$. However, as illustrated in Figure 13, while Student 4 showed some ability to perform algebraic manipulation, there was no evidence of conceptual understanding regarding the vertical stretch of the function.

During the exploratory phase, students investigated how graphs changed when different notations, such as $2f(x)$, $3f(x)$, and $\frac{1}{2}f(x)$, were entered into GeoGebra. The visual feedback from these inputs helped students develop an understanding of vertical stretching and compression.

Figure 14

2f(x) notation explanation

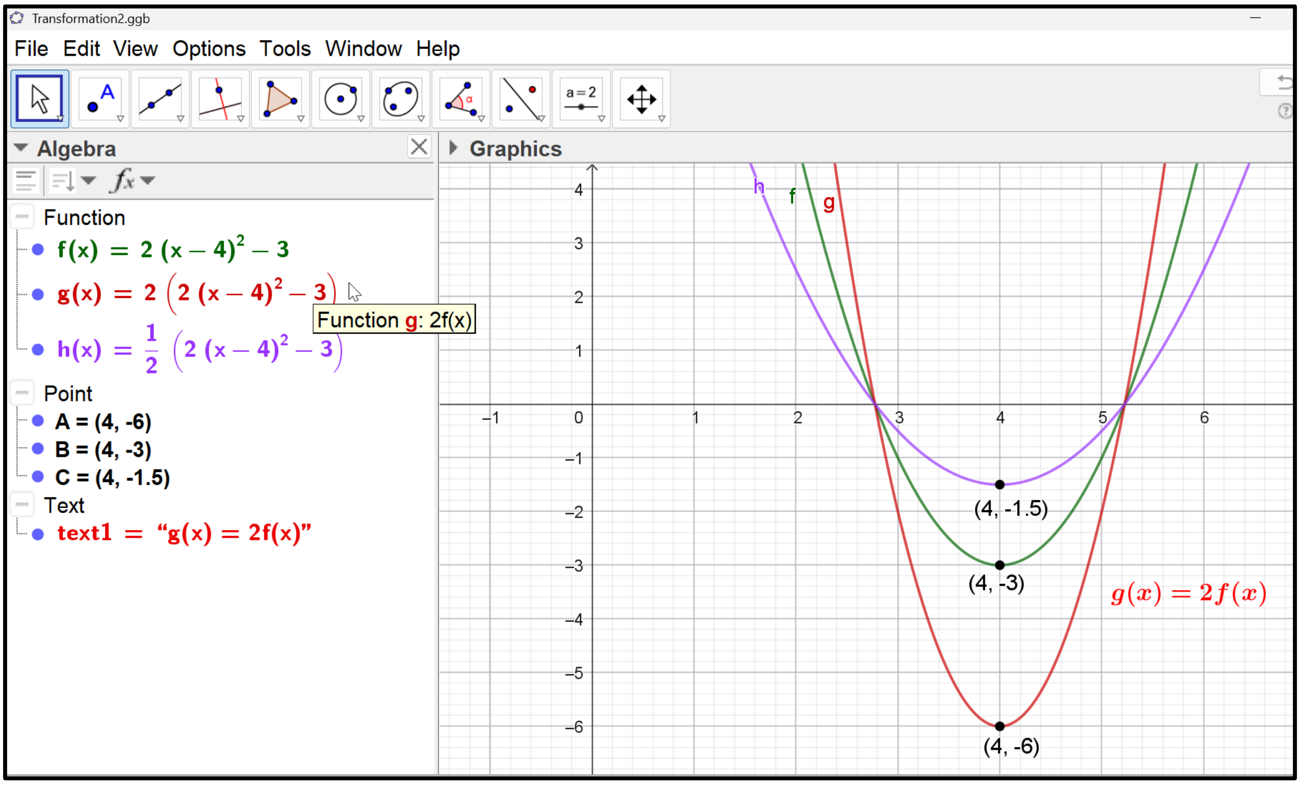
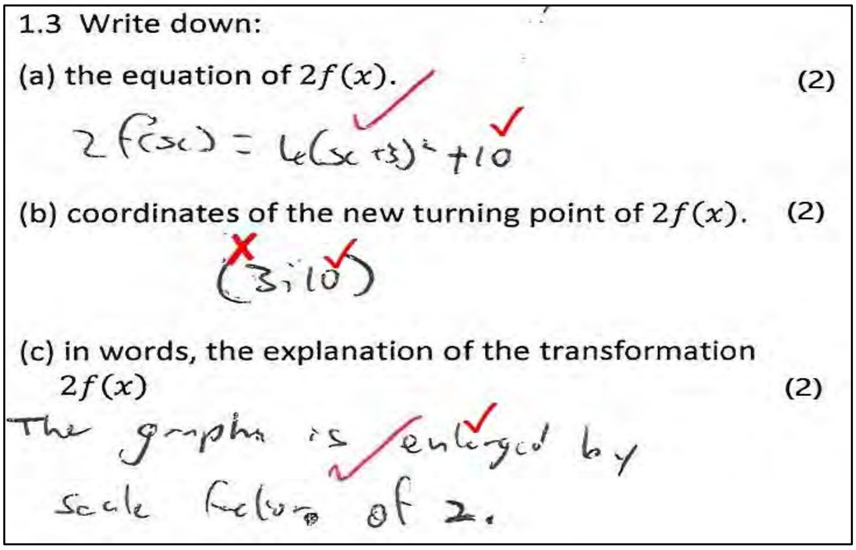


Figure 14 depicts a sample of student responses in the post-test following instruction with GeoGebra. In this task, students actively engaged by entering the equations of functions themselves, allowing them to observe and analyse the impact of algebraic notation on the behaviour of the function $f(x)$. The illustration demonstrates how exposure to dynamic visualisations and interactive manipulation of functions influenced students' ability to apply concepts of transformation and graph interpretation

Figure 15

Student 4's responses to the question on $2f(x)$ in the post-test


1.3 Write down:

(a) the equation of $2f(x)$. (2)

$2f(x) = 6(x+3)^2 + 10$

(b) coordinates of the new turning point of $2f(x)$. (2)

$(3;10)$

(c) in words, the explanation of the transformation $2f(x)$ (2)

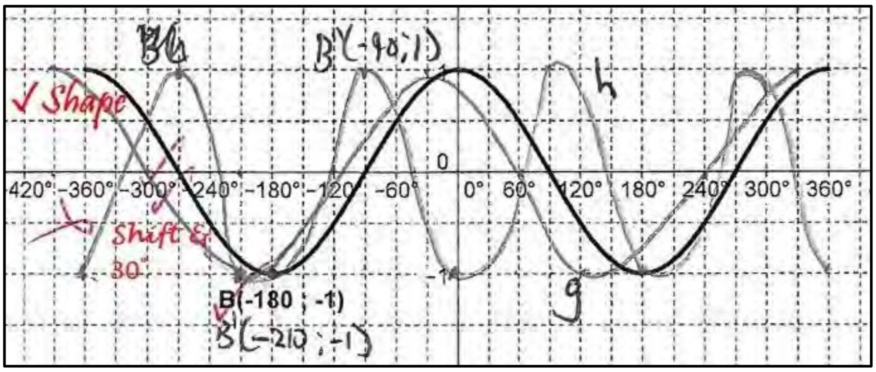
The graph is enlarged by scale factor of 2.

Figure 15 depicts a student's answer to question 1.3 on the post-test, which was nearly precise, except for a single error in the x-coordinate of the turning point, where he stated 3 instead of -3. This seems to be an error, as the student supplied the accurate coordinates for the identical question in the pre-test. The way the students answered this question on the post-test shows how GeoGebra supports the formation of coherence at a deeper level of the notation (Seufert & Brünken, 2004).

Figure 15 shows the same student's pre-test response to sketch the graph of $-2f(x)$ if $f(x) = cos\ x$. It was expected of the student to sketch $f(x + 30°)$ and $-2f(x)$ from the properties of transformations without using a calculator.

Figure 15

Student 4's response to pre-test sketching of $f(x + 30°)$ and $-2f(x)$



As illustrated in Figure 15, the h graph reveals that this student found the notation $-2f(x)$ challenging. In Question 1.3(a) (See Figure 13), the student recognised that y-values must be multiplied by two in the pre-test, but not in the other questions in the same pre-test. This showed that the student recalled the procedure for the transformed function equation, but didn't understand it, and showed comprehension of the notation: $f(x + 30°)$ if $f(x) = cosx$. Figure 16 shows that a student got full marks in post-test question 2a. The student demonstrated superior understanding of this question, which entailed sketching the graph of $-2f(x)$ for $f(x) = cosx$ in the post-test.

Figure 16

Student 4's response to post-test graphic of $f(x + 30°)$ and $-2f(x)$

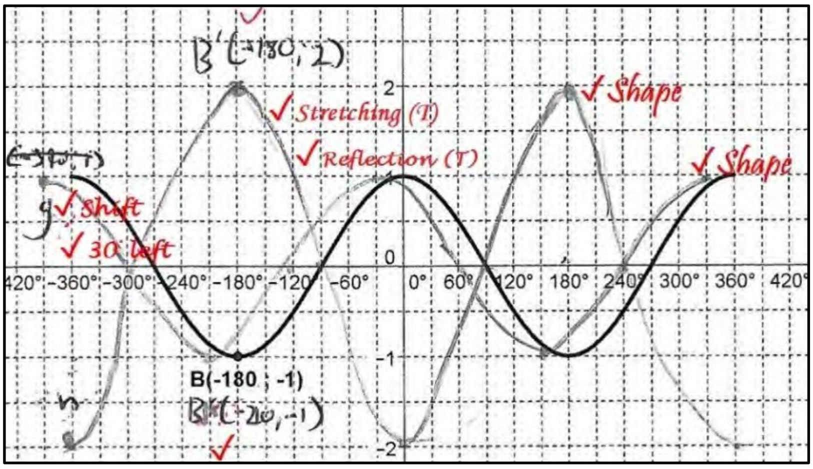


Figure 15 illustrates a student's comprehension of the transformation from $-2f(x)$. The study's pre- and post-tests were written before the introduction of trigonometry. This post-test response indicates that the student understood transformations, as evidenced by his inability to use a calculator to draw the cosine graph or obtain values to plot, and by his lack of formal instruction on how to sketch a trigonometric function graph in SciMathUS. Students were instructed to draw $f(x + 30°)$ and $-2f(x)$ using transformation principles. GeoGebra helped the learner draw this graph. Thus, the learner might correctly use GeoGebra to answer the trigonometric question without a calculator or be taught how to sketch a graph in SciMathUS.

Discussion

The quantitative findings of the study revealed a significance in learning gains in determining the equation of a converted function from algebraic notations (Domain 1), its critical values (turning points, asymptotes, etc.) from (Domain 2), explain the transformation in words from Domain 3, sketch the transformed cosine graph from (Domain 3) and algebraic notations (Domain 4). This finding revealed that students showed learning gains across all domains, with extremely large effects when using GeoGebra. Therefore, the results showed that GeoGebra helps students comprehend function transformations. These findings confirm those of Awaji et al. (2025), who claim that teaching function transformations with GeoGebra is both mathematically and pedagogically beneficial, as transformations are essential to teaching exponential functions. According to Ndlovu (2014), intertwinement makes the mathematics curriculum more coherent. The GeoGebra-focused active-learning intervention also helped low-achieving students (Zulu et al., 2022). This is evident from the large effect size observed in both groups.

The qualitative findings from students' responses during the computer lab observations reflected how GeoGebra may have influenced their attempts. One way they did this was by giving shorter steps on the post-test than on the pre-test. The findings of this study clearly revealed that the algebraic and graphical windows of GeoGebra helped visualise the transformations of functions and graphs, aiding their comprehension. This finding strengthens the proposition that GeoGebra's visualisation functionalities enhance learning (Baccaglini-Frank *et al.*, 2020; Ndlovu et al., 2011). This finding is corroborated by Velichová (2021) and Svitek et al. (2022), who emphasise that visual dynamics open the way to discover connections and understand mutual dependencies, which may often be more important than detailed, fragmented knowledge.

Additionally, the tasks that required students to use GeoGebra to sketch the graphs of the trigonometric functions were intended to teach them that all transformations are the same for every graph and that translation and reflection refer to the same graph. More students were able to illustrate the cosine function by simply comprehending the ideas of translation, reflection, and dilation after being exposed to GeoGebra. Student responses showed that GeoGebra improved their physical and logico-mathematical understanding of $-f(x)$, a$f(x)$, and f(x+a) . This demonstrated how students transitioned from informal to formal thinking, or from horizontal to vertical thinking. These results may be due to a greater emphasis on input and output values when learning function transformations in the program to solve related problems. Mokotjo (2020) also reiterated that transformations of functions with input and output values should be taught using technologically enhanced tooling because students are occasionally given a technique to follow, which is a challenge that GeoGebra solves seamlessly. During the GeoGebra intervention, students were able to manipulate a point and subsequently determine the new equation of the transformed graph. They could connect algebraic and graphical representations. Students investigated, using GeoGebra, that the notation $af(x)$, where $a \in \mathbb{R}$, denotes identical transformations for any function. The students could thus associate the type of transformation with the notation, thereby enhancing their logico-mathematical knowledge through this method.

Additionally, the qualitative evaluations of pre- and post-test scripts for the transformation of functions corroborate this assumption, as students answered differently on the post-test, indicating that they understood transformations. More students answered higher-cognition questions. During the post-test, students demonstrated how GeoGebra improved structure coherence by answering questions about on $-f(x)$, $2f(x)$, $-2f(x)$ and $f(x+30^{\circ})$. The students began using shorter approaches to answer questions. They show understanding of different notations and multiple representations in GeoGebra, which helps with a deeper understanding of transformations (Seufert & Brünken, 2004). This finding corroborates earlier findings that integrating GeoGebra into the teaching of mathematics positively affected students' understanding, being considered an effective technology integration in learning (Martinez, 2017).

Furthermore, the findings showed that some students used sketching to solve the posed mathematical problems. This activity is important as teachers are required to explain why procedures function rather than how they work when teaching function transformations (Hatisaru, 2023). This method allowed students to determine the crucial values (turning points, asymptotes, etc.) of the transformed cosine graph from algebraic notation. The findings also confirm Juandi et al. (2025)'s premise that GeoGebra enhances structural and functional comprehension. GeoGebra sliders characterise the behaviour of function graphs as their algebraic form changes. Learning with GeoGebra also fostered students interact with each other and the teacher. Boles (2024) asserts that students can retain 90% of information within 24 hours, provided they can interact while learning, solve complex problems, and make GeoGebra instructional design acceptable to students. This finding aligned with that of Dahal et al. (2019) and Serpa (2024), whose findings show that students' active participation in learning aided transformative pedagogy. This finding could be due to the dynamic GeoGebra environment, which aided collaboration (Aksu, N., & Zengin, 2024).

The paired t-test (in Table 1) compares students' means on the pre- and post-tests across the different domains. The analysis of effect sizes across the four domains in the pre- and post-tests reveals substantial learning gains attributable to GeoGebra-based instruction. Effect sizes ranged from large to exceptionally large, underscoring the pedagogical significance of dynamic visualisation tools in mathematics education. Domains 1, 2, and 3 demonstrated very large effects, while Domain 4 exhibited a large effect, albeit comparatively smaller.

Specifically, Domain 1 yielded an effect size of 1.69 (Cohen's d), classified as extremely large, indicating a pronounced practical impact on students' ability to derive the equation of a transformed graph from given algebraic representations. Similarly, Domain 2 recorded an effect size of 2.05, reflecting substantial improvement in students' proficiency in determining critical values — such as turning points and asymptotes — of transformed functions. Domain 3, with an effect size of 1.95, suggests significant advancement in students' capacity to articulate transformations verbally based on algebraic notation. Although Domain 4 yielded a lower effect size of 0.95, it still indicates a meaningful improvement in students' ability to sketch transformed cosine graphs from algebraic expressions.

The consistently high effect sizes across domains provide compelling evidence that integrating multiple representations — algebraic and graphical — through GeoGebra fosters deeper conceptual understanding and measurable learning gains (Seufert & Brünke, 2004). Furthermore, qualitative analyses of students' responses corroborate these quantitative findings, reinforcing the conclusion that GeoGebra facilitates higher-order comprehension of function transformations, including shifts, reflections, and stretches. This triangulation of evidence substantiates the pedagogical efficacy of dynamic mathematics software in promoting conceptual mastery within the domain of function transformations.

Several challenges emerged during the instructional sessions, as both the researcher and the students reported during interviews. One of the most prominent difficulties observed was zooming in and out in GeoGebra. During the session on function transformations, some students inadvertently scrolled the mouse wheel, causing the graph to shift out of view. As a result, they struggled to locate key features such as the turning point of the transformed function (see the red graph in Figure 5.85). Despite an initial orientation on navigating GeoGebra, students encountered further challenges in locating specific icons and tools within the software interface. Variability in computer literacy also played a significant role. While 83% of students had prior experience with a computer, 17% had never used one before, which affected their confidence and efficiency during the activity. Another recurring issue involved correctly typing the equations of functions. For some students, entering expressions — particularly those involving exponents — proved difficult and occasionally led to errors in constructing the intended transformations.

Conclusions

Based on the findings of this study, the triangulation of quantitative and qualitative results reflected students' learning gains using GeoGebra, helping them understand how functions changed over time. This is based on qualitative scripts and quantitative analysis. Students value experienced context and direction, and appreciate exploration to effectively learn about the transformation of functions and graphs, with statistically significant effects and large effect sizes. As such, GeoGebra aided students' understanding and abstraction of function transformations, which would help address the DBE's (2013, 2014, 2015c) concerns about the lack of understanding of function behaviour in NSC Mathematics and students' conceptual comprehension issues. Therefore, this study recommends that mathematics teachers incorporate GeoGebra into their teaching and learning resources.

Suggestions for Future Research

Future research should explore the integration of GeoGebra across additional Grade 12 Mathematics topics, such as Circle Geometry, Trigonometry, and Differential Calculus, using a true experimental design with a control group. The control group can be taught through traditional methods, such as rote memorisation or paper-based conceptual activities, while the experimental group engages with dynamic representations in GeoGebra. Building on Pfeiffer's (2017) study, which demonstrated the potential of GeoGebra in teaching function transformations and Circle Geometry primarily through teacher-led projection, future studies should investigate student-centred interaction over extended periods and across diverse domains. Cognitive analyses of how learners interpret and coordinate multiple representations (algebraic, graphical, symbolic) within GeoGebra are recommended, as these processes underpin conceptual understanding and exploratory reasoning. Furthermore, longitudinal research tracking the same cohort over multiple time points (e.g., immediately after the intervention and six months later) would provide valuable insights into the durability of learning gains, the transfer of knowledge to new contexts, and the development of higher-order mathematical thinking.

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Conflict of Interest

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